# Algorithms and Data Structures

## C5. Shortest Paths: Foundations

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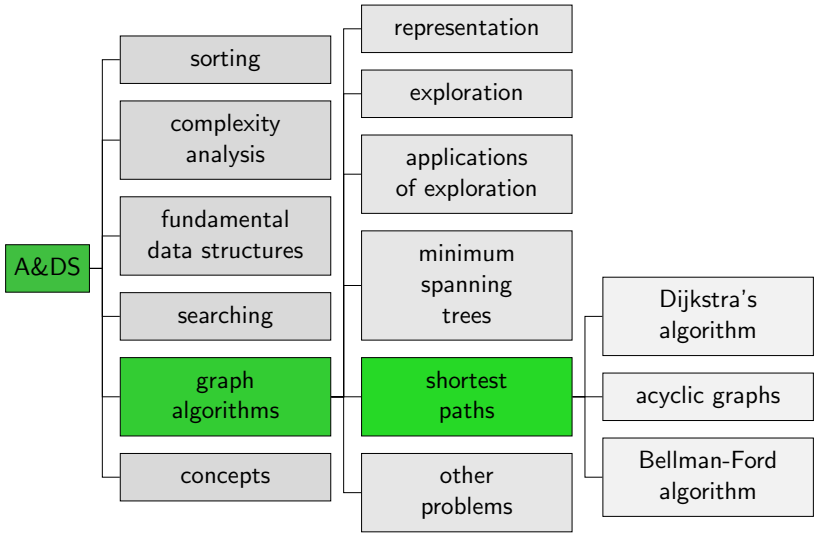
## Content of the Course

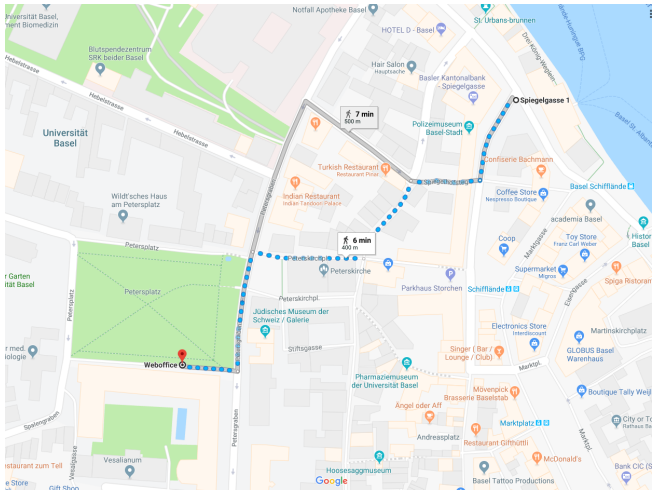







# C5.1 Introduction

## Google Maps








## Seam Carving





## Applications

- Route planning
- Path planning in games
- robot navigation
- seam carving
- automated planning
- typesetting in TeX
- routing protocols in networks (OSPF, BGP, RIP)
- routing of telecommunication messages
- traffic routing

Source (partially): Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993





## Variants

What are we interested in?

- Single source: from one vertex $s$ to all other vertices
- Single sink: from all vertices to one vertex $t$
- Source-sink: from vertex $s$ to vertex $t$
- All pairs: from every vertex to every vertex

Graph properties

- arbitrary / non-negative / Euclidean weights
- arbitrary / non-negative / no cycles

# C5.2 Foundations

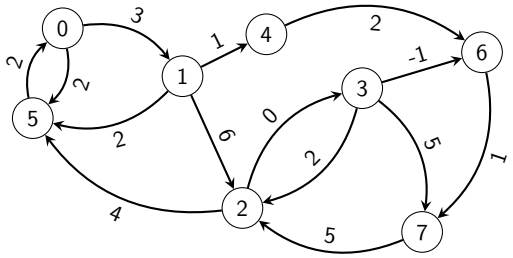




## Weighted Directed Graphs

Same (high-level) definition of weighted graphs as before, but now we consider directed graphs.

**Directed Graph**
An (edge)-weighted graph associates every edge $e$ with a weight (or cost) $weight(e) \in \mathbb{R}$.




Reminder: A directed graphs is also called a digraph.





## API for Weighted Directed Edge

```
1  class DirectedEdge:
2    # Edge from n1 to n2 with weight w
3    def __init__(n1: int, n2: int, w: float) -> None
4
5    # Weight of the edge
6    def weight() -> float
7
8    # Initial vertex of the edge
9    def from_node() -> int
10
11   # Terminal vertex of the edge
12   def to_node() -> int
```





## API for Weighted Digraphs

```
1  class EdgeWeightedDigraph:
2    # Graph with no_nodes vertices and no edges
3    def __init__(no_nodes: int) -> None
4
5    # Add weighted edge
6    def add_edge(e: DirectedEdge) -> None
7
8    # Number of vertices
9    def no_nodes() -> int
10
11   # Number of edges
12   def no_edges() -> int
13
14   # All outgoing edges of n
15   def outgoing_edges(n: int) -> Generator[DirectedEdge]
16
17   # All edges
18   def all_edges() -> Generator[DirectedEdge]
```

# Shortest Path Problem

**Single-source shortest path problem, SSSP**

- Given: Graph and start vertex $s$
- Query for vertex $v$
  - Is there a path from $s$ to $v$?
  - If yes, what is the shortest path?
- In weighted graphs:
  Shortest path is the one with lowest weight
  (= minimal sum of edge costs)





# API for Shortest-path Implementation

The algorithms for shortest paths should implement the following interface:

```
1  class ShortestPaths:
2    # Initialization for start vertex s
3    def __init__(graph: EdgeWeightedDigraph, s: int) -> None
4
5    # Distance from s to v; infinity, if there is no path
6    def dist_to(v: int) -> float
7
8    # Is there a path from s to v?
9    def has_path_to(v: int) -> bool
10
11   # Path from s to v; None, if there is none
12   def path_to(v: int) -> Generator[DirectedEdge]
```



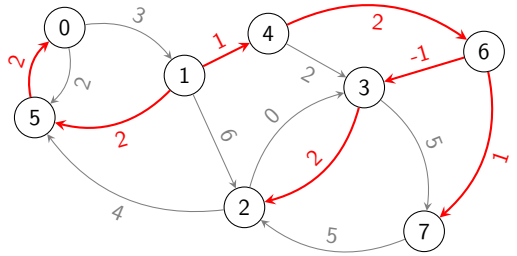


# Shortest-path Tree

**Shortest-path Tree**

For a weighted digraph $G$ and vertex $s$, a shortest-path tree is a subgraph that

- forms a directed tree with root $s$,
- contains all vertices that are reachable from $s$, and
- for which every path in the tree is a shortest path in $G$.





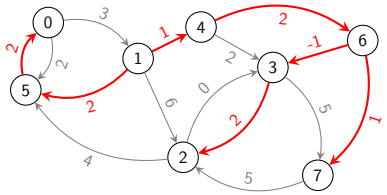



# Shortest-path Tree: Representation

Representation: arrays indexed by vertex

- `parent` with reference to parent vertex
  None for unreachable vertices and start vertex
- `distance` with distance from the start vertex
  $\infty$ for unreachable vertices

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| parent | 5 | None | 3 | 6 | 1 | 1 | 4 | 6 |
| distance | 4 | 0 | 4 | 2 | 1 | 2 | 3 | 4 |




What about parallel edges?

## Extracting Shortest Paths

```
1   def path_to(self, node):
2       if self.distance[node] == float('inf'):
3           yield None
4       elif self.parent[node] is None:
5           yield node
6       else:
7           # output path from start to parent node
8           self.path_to(self.parent[node])
9           # finish with node
10          yield node
```

This implementation generates a sequence of vertices. What do we have to change to generate a corresponding sequence of edges?





## Relaxation

Relaxing edge $(u, v)$

- `distance[u]`: cost of the shortest known path to $u$
- `distance[v]`: cost of the shortest known path to $v$
- `parent[v]`: predecessor of v in the shortest known paths to $v$
- Does edge $(u, v)$ establish a shorter path to $v$ (through $u$)?
- If yes, update `distance[v]` and `parent[v]`.

Illustration: Whiteboard





## Relaxation

```
1   def relax(self, edge):
2       u = edge.from_node()
3       v = edge.to_node()
4       if self.distance[v] > self.distance[u] + edge.weight():
5           self.parent[v] = u
6           self.distance[v] = self.distance[u] + edge.weight()
```





# C5.3 Optimality Criterion and Generic Algorithm

## Optimality Criterion

**Theorem**

*Let $G$ be a weighted digraph without negative cycles.*
*Array `distance[]` contains the cost of the shortest paths from $s$ if and only if*

1. `distance[s]` $= 0$
2. `distance[`$w$`]` $\leq$ `distance[`$v$`]` $+$ *weight*$(e)$ *for all edges* $e = (v, w)$, *and*
3. *for all vertices* $v$, `distance[`$v$`]` *is the cost of some path from* $s$ *to* $v$, *or* $\infty$ *if there is no such path.*





## Optimality Criterion (Continued)

**Proof**

"$\Rightarrow$"

1. Since the graph has no cycles of negative cost, no path from $s$ to $s$ can have negative cost. Thus, the empty path is optimal and `distance[s]` is 0.
2. Consider an arbitrary edge $e$ from $u$ to $v$.
   The shortest path from $s$ to $u$ has cost `distance[u]`. If we extend this path by edge $e$, we have a path from $s$ to $v$ of cost `distance[u]` $+$ *weight*$(e)$. Thus, the cost of a shortest path from $s$ to $v$ cannot be larger and it holds that `distance[v]` $\leq$ `distance[u]` $+$ *weight*$(e)$.
3. Trivially true.

...





## Optimality Criterion (Continued)

**Proof (continued).**

"$\Leftarrow$"

For unreachable vertices, the value is infinity by definition.

Consider an arbitrary vertex $v$ and a shortest path $p = (v_0, \ldots, v_n)$ from $s$ to $v$, i.e. $v_0 = s$, $v_n = v$.
For $i \in \{1, \ldots, n\}$, let $e_i$ be a cheapest edge from $v_{i-1}$ to $v_i$.
Since all inequalities are satisfied, we have

$$\begin{aligned} \texttt{distance}[v_n] &\leq \texttt{distance}[v_{n-1}] + \mathit{weight}(e_n) \\ &\leq \texttt{distance}[v_{n-2}] + \mathit{weight}(e_{n-1}) + \mathit{weight}(e_n) \\ &\leq \ldots \leq \mathit{weight}(e_1) + \cdots + \mathit{weight}(e_n) \\ &= \text{ cost of an optimal path.} \end{aligned}$$

Due to 3, `distance`$[v_n]$ cannot be lower than the optimal cost. □





## Generic Algorithm

**Generic Algorithm for Start Vertex $s$**

- Initialize `distance`$[s] = 0$ and `distance`$[v] = \infty$ for all other vertices
- As long as the optimality criterion is not satisfied: Relax an arbitrary edge

Correct:

- Finite `distance[v]` always corresponds to the cost of a path from s to v.
- Every successful relaxation reduces `distance[v]` for some v.
- For every vertex, the distance can only be reduced finitely often.

# C5.4 Summary





## Summary

- Single-source shortest paths: Compute in a weighted digraph the shortest paths from a given vertex to all reachable vertices.
- Relaxation: If for edge (u,v) the best known distance to v is larger than the one to u plus the edge cost, then update the distance to v (with predecessor u).
- Generic algorithm
  - Based on relaxation and optimality criterion.
  - Every instantiation is correct for all weighted digraphs without negative-cost cycles.
  - Specific instantiations: next chapter.