# Algorithms and Data Structures

## C3. Disjoint-set Data Structure/Union-Find

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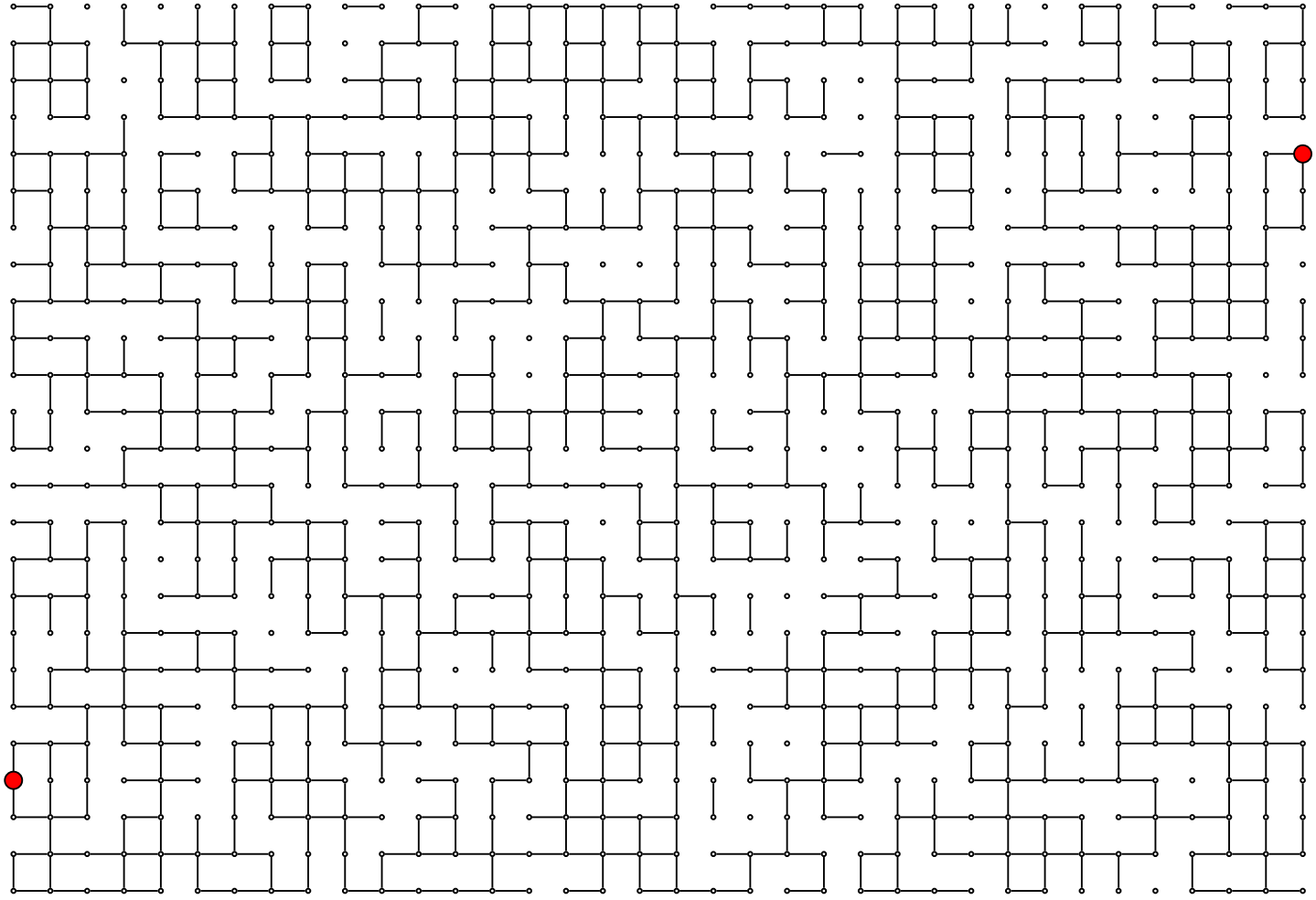
# Union-Find

# Questions



Are the red vertices connected?

How many connected components does the graph have?

# Connected Components as Disjoint Sets

Set of conn. components as collection of disjoint sets of objects.

- One set for all vertices of one connected component.
- Operations:
  - Union: Given two objects, merge the sets that contain them into one.
    Introduce a new edge between the given vertices, connecting their connected components.
  - Find: Given an object, return a representative of the set that contains it.
    Given a vertex, return a representative vertex for its connected component.
    - Must return the same representative for all objects in the set.
    - The representative may only change if set gets merged.
    - Two objects are in the same set (two vertices are connected) if find returns the same representative for them.
  - Count: Return the number of sets
    Return the number of connected components.

# Union-Find Data Type

```
1   class UnionFind:
2     # Initialization for n objects (with names 0, ..., n-1).
3     def __init__(n: int) -> None
4
5     # Merge the sets containing objects v and w.
6     def union(v: int, w: int) -> None
7
8     # Representative for set containing v.
9     # May change if set is merged by call of union,
10    # but not otherwise.
11    def find(v: int) -> int
12
13    # Number of sets.
14    def count() -> int
```

# (Somewhat) Naive Algorithm: Quick-Find

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- Entry at position $i$ is representative of the set containing $i$.
- Initially, every object is (alone) in its own set, and thus its representative.
- Update the array in every call of `union`.

# Quick-Find Data Structure

```
1  class QuickFind:
2      def __init__(self, no_nodes):
3          self.components = no_nodes
4          self.representative = list(range(no_nodes))
5
6      def count(self):
7          return self.components
8
9      def find(self, v):
10         return self.representative[v]
```

[0, 1, ..., no_nodes-1]

# Quick-Find Data Structure (Continued)

```
20      def union(self, v, w):
21          repr_v = self.find(v)
22          repr_w = self.find(w)
23          if repr_v == repr_w:  # already in same component
24              return
25          # replace all occurrences of repr_v in
26          # self.representative with repr_w
27          for i in range(len(self.representative)):
28              if self.representative[i] == repr_v:
29                  self.representative[i] = repr_w
30          self.components -= 1  # we merged two components
```

---

Running time?

- Cost model = number of array accesses
- one access for every call of `find`
- between and accesses
  for every call of `union` that merges two components

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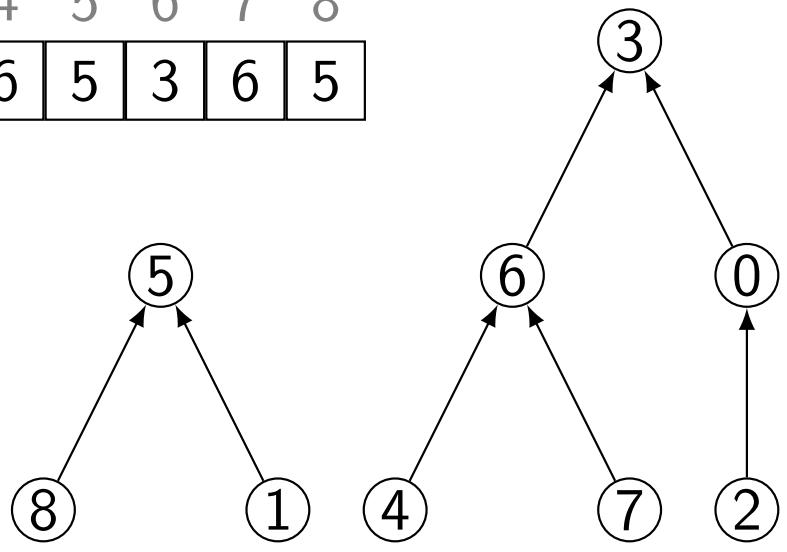
- one access for every call of `find`
- between $n + 3$ and $2n + 1$ accesses
  for every call of `union` that merges two components

# Better: Quick-Union aka Disjoint-set Forest

- (implicit) tree for representing each set
- represented as array with parent nodes as entries (root: reference to itself)

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 0 | 3 | 6 | 5 | 3 | 6 | 5 |



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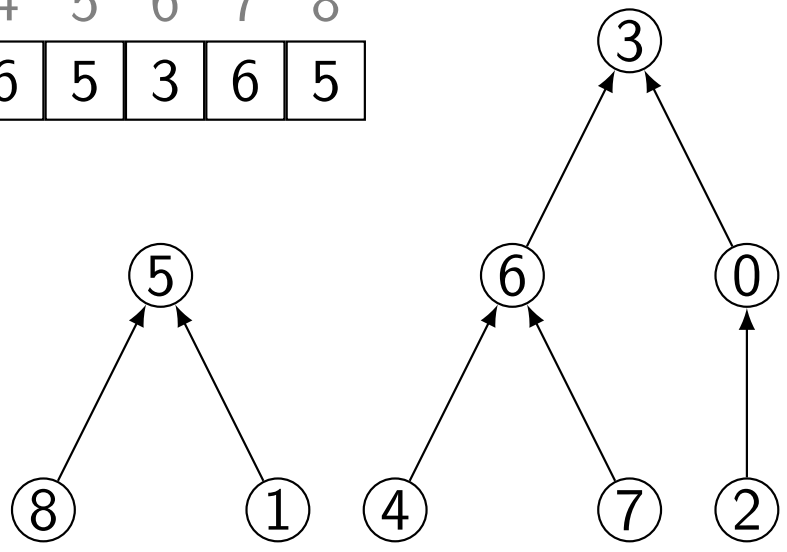


- Root node serves as representative of the set.

# Quick-Union Data Structure

```
1  class QuickUnion:
2      def __init__(self, no_nodes):
3          self.parent = list(range(no_nodes))
4          self.components = no_nodes
5
6      def find(self, v):
7          while self.parent[v] != v:
8              v = self.parent[v]
9          return v
10
11     def union(self, v, w):
12         repr_v = self.find(v)
13         repr_w = self.find(w)
14         if repr_v == repr_w:  # already in same component
15             return
16         self.parent[repr_v] = repr_w
17         self.components -= 1
18
19     # count as in QuickFind
```

# First Improvement

- Problem with Quick-Union: Trees can degenerate into chains.
  $\rightarrow$ `find` requires linear time in the size of the set.
- Idea: In union the root of the tree with lower height becomes a child of the root of the higher tree.

# Ranked Quick-Union Algorithm

```
1  class RankedQuickUnion:
2      def __init__(self, no_nodes):
3          self.parent = list(range(no_nodes))
4          self.components = no_nodes
5          self.rank = [0] * no_nodes  # [0, ..., 0]
6
7      def union(self, v, w):
8          repr_v = self.find(v)
9          repr_w = self.find(w)
10         if repr_v == repr_w:
11             return
12         if self.rank[repr_w] < self.rank[repr_v]:
13             self.parent[repr_w] = repr_v
14         else:
15             self.parent[repr_v] = repr_w
16             if self.rank[repr_v] == self.rank[repr_w]:
17                 self.rank[repr_w] += 1
18         self.components -= 1
19
20     # connected, count and find as in QuickUnion
```

# Second Improvement

## Path Compression

- Idea: During find, reconnect all traversed nodes to the root.
- We do not update the height of the tree during path compression.
  - Value of `rank` can deviate from the actual height.
  - That's why it is called rank and not height.

# Ranked Quick-Union Algorithm with Path Compression

```
1  class RankedQuickUnionWithPathCompression:
2      def __init__(self, no_nodes):
3          self.parent = list(range(no_nodes))
4          self.components = no_nodes
5          self.rank = [0] * no_nodes  # [0, ..., 0]
6
7      def find(self, v):
8          if self.parent[v] == v:
9              return v
10         root = self.find(self.parent[v])
11         self.parent[v] = root
12         return root
13
14     # connected, count and union as in RankedQuickUnion
```

# Discussion

- With all improvements, we achieve almost constant amortized cost for all operations.

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- More precisely: [Tarjan 1975]
  - $m$ calls of `find` for $n$ objects (and at most $n-1$ calls of `union`, merging two components)
  - $O(m\alpha(m,n))$ array accesses
  - $\alpha$ is inverse of a variant of the Ackermann function
  - In practice $\alpha(m,n) \leq 3$.

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  - $O(m\alpha(m,n))$ array accesses
  - $\alpha$ is inverse of a variant of the Ackermann function
  - In practice $\alpha(m,n) \leq 3$.
- Nevertheless: there cannot be a union-find structure that guarantees linear running time.
  (under cell-probe model, only accounting for memory access)

# Comparison to Exploration-based Approach

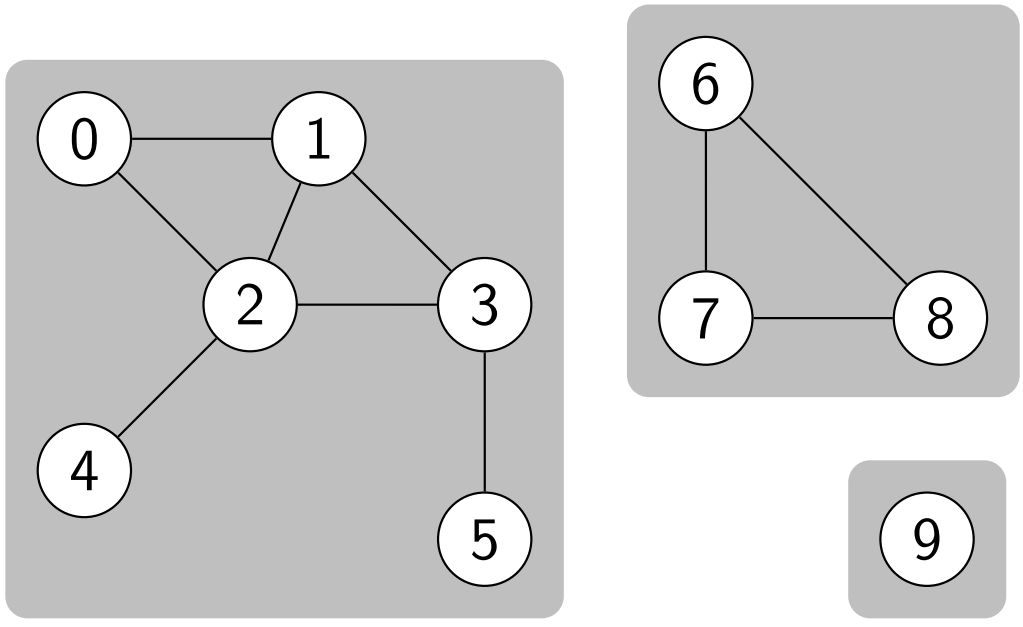
- Chapter C2: Algorithm `ConnectedComponents`, based on graph exploration.
- After the precomputation, queries only require constant time.
- In practise, disjoint-set forests are often faster, because for many applications, we do not have to build up the full graph.
- If the graph has already been built up, graph exploration can be better.
- Another advantage of union find:
  - Dynamic approach
  - We can easily introduce further edges.

# Connected Components and Equivalence Classes

# Reminder: Connected Components

Undirected graph

- Two vertices $u$ and $v$ are in the same connected component if there is a path between $u$ and $v$ (= vertices $u$ and $v$ are connected).



# Connected Components: Properties

- The connected components define a partition of the vertices:
  - Every vertex is in a connected component.
  - No vertex is in more than one connected component.
- "is connected with" is an equivalence relation.
  - reflexive: Every vertex is connected with itself.
  - symmetric: If $u$ is connected with $v$, then $v$ is connected with $u$.
  - transitive: If $u$ is connected with $v$, and $v$ with $w$, then $u$ is connected with $w$.

# Partition in General

## Definition (Partition)

A partition of a finite set $M$ is a set $P$ of non-empty subsets of $M$, such that

- every element of $M$ is in some set in $P$: $\bigcup_{S \in P} S = M$, and
- that sets in $P$ are pairwise disjoint: $S \cap S' = \emptyset$ for $S, S' \in P$ with $S \neq S'$.

The sets in $P$ are called blocks.

$M = \{e_1, \ldots, e_5\}$

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# Equivalence Relations in General

**Definition (Equivalence Relation)**

An equivalence relation over set $M$ is a
symmetric, transitive and reflexive relation $R \subseteq M \times M$.
We write $a \sim b$ for $(a, b) \in R$ and say that $a$ is equivalent to $b$.

- symmetric: $a \sim b$ implies $b \sim a$
- transitive: $a \sim b$ and $b \sim c$ implies $a \sim c$
- reflexive: for all $e \in M$: $e \sim e$

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- Vice versa:
  For partition $P$ define $R = \{(x, y) \mid \exists B \in P : x, y \in B\}$
  (i.e. $x \sim y$ if and only if $x$ and $y$ are in the same block).
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  (i.e. $x \sim y$ if and only if $x$ and $y$ are in the same block).
  Then $R$ is an equivalence relation.
- We can consider blocks in partitions as equivalence classes and vice versa.

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  - no "unnecessary" equivalences.

Can use union-find data structures to determine equivalence classes.

# Summary

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- A union-find data structure maintains a collection of disjoint sets.
  - union: merge two sets.
  - find: identify the set containing an object and return its representative.
- Good implementation: Disjoint-set forest
  with improvements to keep the height of the trees low:
  - Union adjoins the shorter tree to the taller tree.
  - Find reconnects traversed nodes to the root
    (path compression).
- Applications:
  - Connected components
  - Finest equivalence relation