

# Algorithms and Data Structures

## C1. Graphs: Foundations and Exploration

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April 23, 2026

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# Algorithms and Data Structures

April 23, 2026 — C1. Graphs: Foundations and Exploration

C1.1 Motivation

C1.2 Definition

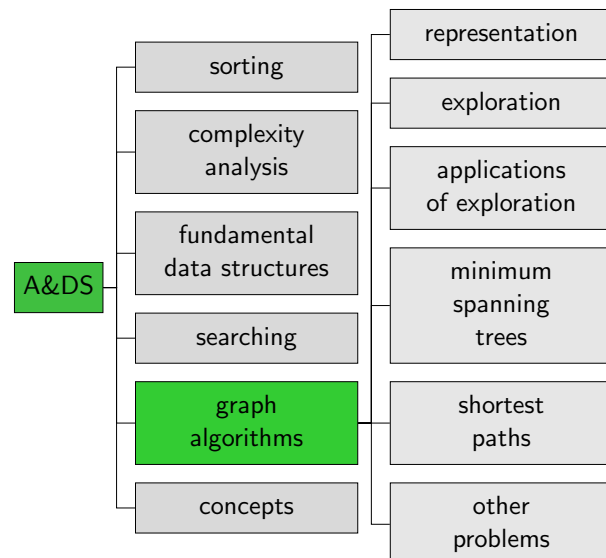
C1.3 Representation

C1.4 Graph Exploration

C1.5 Summary

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## Content of the Course



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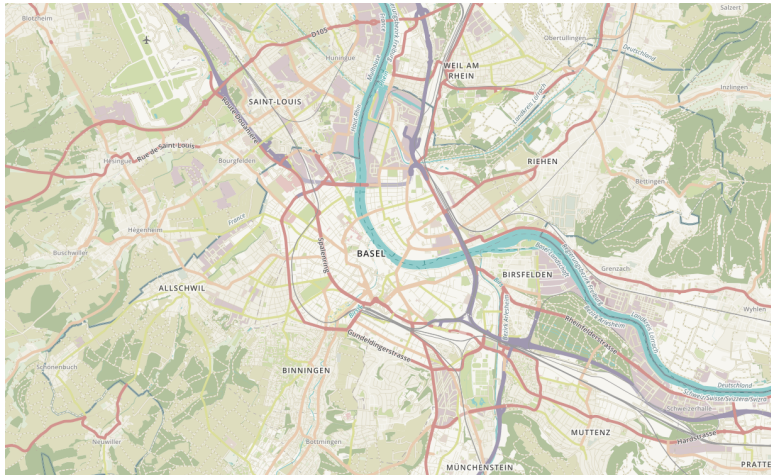
C1. Graphs: Foundations and Exploration

Motivation

## C1.1 Motivation

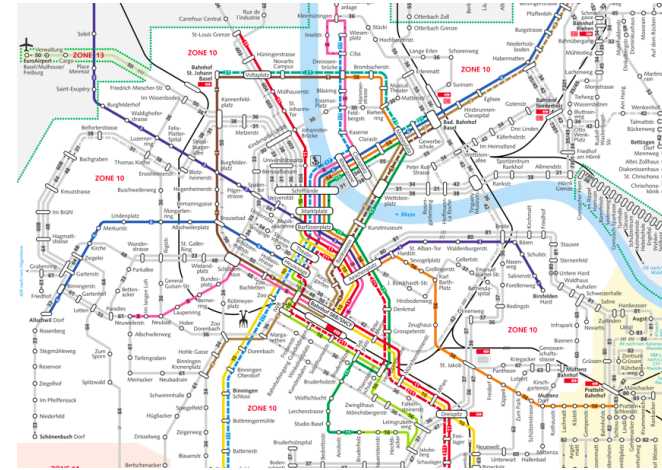
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# Street Maps



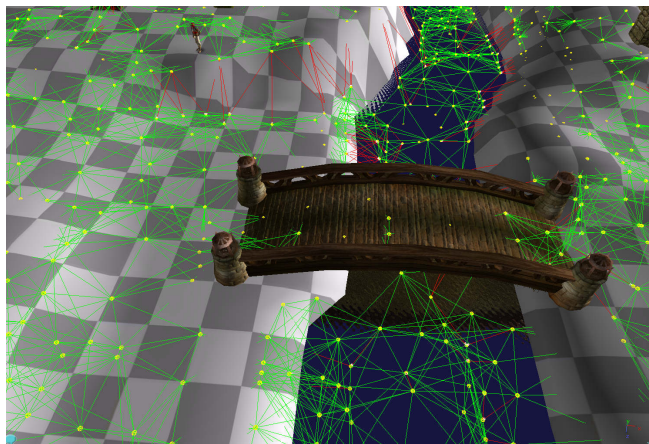
openstreetmap.org

# Route Networks



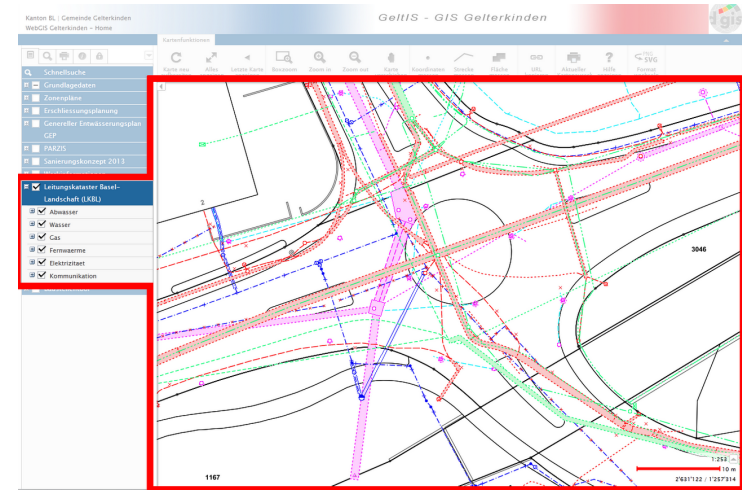
tnw.ch

# Navigation Networks in Games



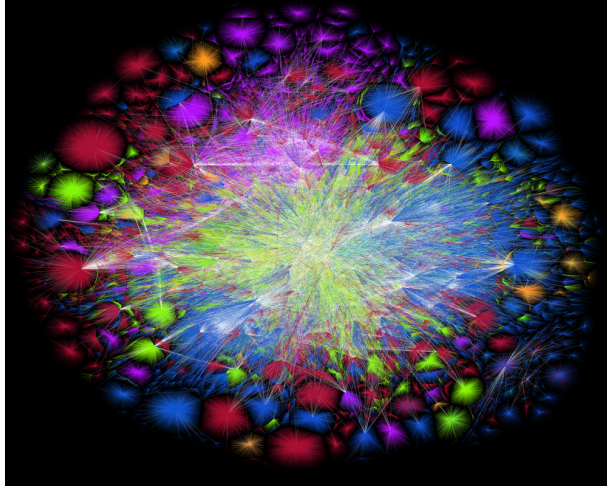
heroengine.com

# Urban Supply System



dgis.info

## Internet



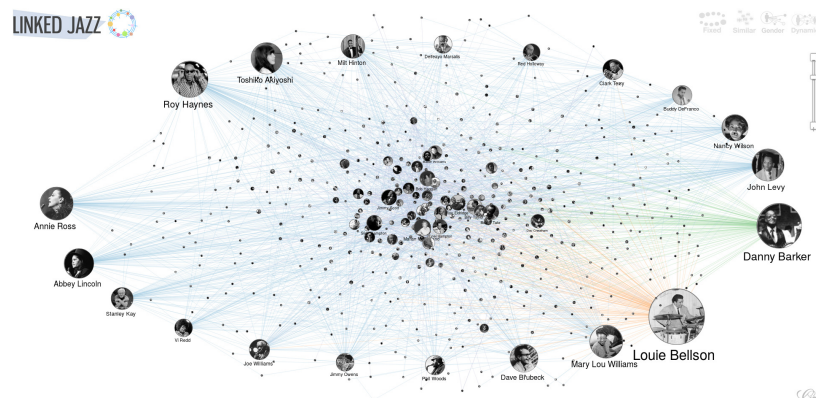
Barrett Lyon / The Opte Project  
Visualization of the routing paths of the Internet.

## Social Networks



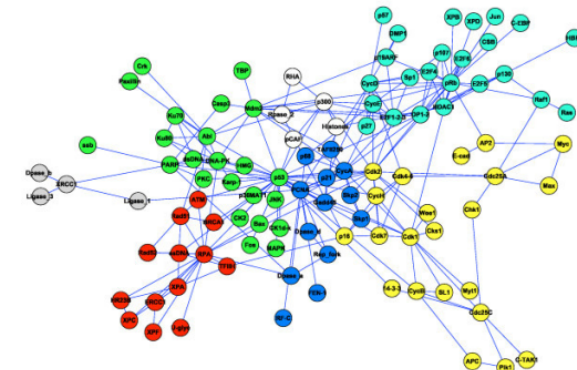
"Visualizing Friendships" by Paul Butler

## Collaboration



linkedjazz.org

## Protein Interaction



Network representation of the p53 protein interactions  
Module detection in complex networks using integer optimisation,  
Xu G, Bennett L, Papageorgiou LG, Tsoka S - Algorithms Mol Biol (2010)

## Possible Questions

- ▶ Are A and B connected?
- ▶ What is the shortest connection between A and B?
- ▶ What is the longest distance between two elements?
- ▶ How much water can the sewer system discharge?

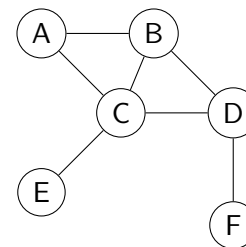
## Abstract Graphs

A **Graph** consists of **vertices** and **edges** between vertices.

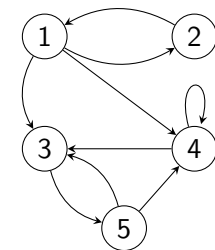
|          | Vertices                 | Edges          |
|----------|--------------------------|----------------|
| Streets  | Crossing                 | Street segment |
| Internet | AS ( $\approx$ Provider) | Route          |
| Facebook | Person                   | Friendship     |
| Proteins | Protein                  | Interaction    |

## C1.2 Definition

## Undirected and Directed Graphs



undirected graph



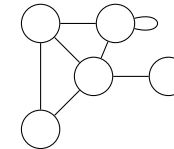
directed graph

## Graphs

- ▶ A graph is a pair  $(V, E)$  comprising
  - ▶  $V$ : finite set of vertices
  - ▶  $E$ : finite set of edges
- ▶ Every edge connects two vertices  $u$  and  $v$ 
  - ▶ undirected graph: set  $\{u, v\}$
  - ▶ directed graph: pair  $(u, v)$
- ▶ **Multigraphs** permit multiple parallel edges between the same nodes.
- ▶ **Weighted** graphs associate each edge with a weight (a number).

## Undirected Graphs: Terminology

- ▶ **Neighbours** of a vertex  $u$ : all vertices  $v$  with  $\{u, v\} \in E$ .
- ▶  $\text{degree}(v)$ : **Degree** of a vertex = **Number of neighbours**.
  - ▶ Exception: **Self-loops** increase the degree by 2.  
Self-loop = edge that connects a vertex with itself.

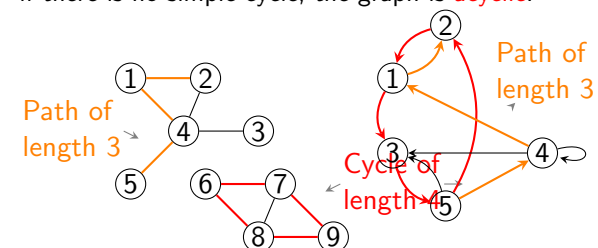


## Directed Graphs: Terminology

- ▶ **Successors** of vertex  $u$ : all vertices  $v$  with  $(u, v) \in E$ .
- ▶ **Predecessors** of vertex  $u$ : all vertices  $v$  with  $(v, u) \in E$ .
- ▶  $\text{outdegree}(v)$ : **outdegree** = number of **successors**
- ▶  $\text{indegree}(v)$ : **indegree** = number of **predecessors**

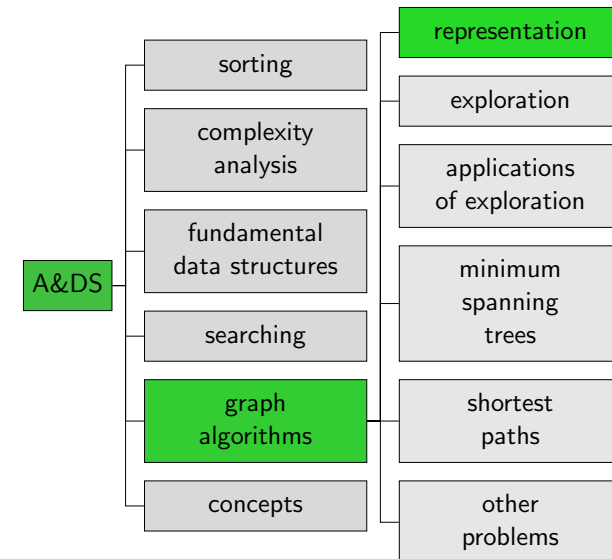
## Paths and Cycles

- ▶ **Path of length  $n$** : Sequence  $(v_0, \dots, v_n)$  of vertices with
  - ▶  $\{v_i, v_{i+1}\} \in E$  for  $i = 0, \dots, n-1$  (undirected graph)
  - ▶  $(v_i, v_{i+1}) \in E$  for  $i = 0, \dots, n-1$  (directed graph)
  - ▶ The path is **simple** if all vertices are distinct.
  - ▶ Example:  $(5, 4, 1, 2)$
- ▶ **Cycle**: Path with equal start and end vertex ( $v_0 = v_n$ ) of length  $> 0$ .
  - ▶  $(6, 7, 9, 8, 6)$  in the undirected and  $(5, 2, 1, 3, 5)$  in the directed example graph
  - ▶ The cycle is **simple** if all vertices  $v_1, \dots, v_n$  are distinct.
  - ▶ if there is no simple cycle, the graph is **acyclic**.



## C1.3 Representation

## Content of the Course



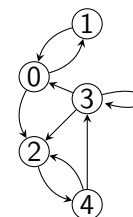
## Representation of Vertices

- ▶ We use numbers 0 to  $|V| - 1$  for the vertices.
- ▶ If not the case in application: Use a map to convert from names to numbers.

## Adjacency-matrix Representation

Graph  $G = (\{0, \dots, |V| - 1\}, E)$  represented as  $|V| \times |V|$  matrix with entries  $a_{ik}$  (in row  $i$ , column  $k$ ):

$$a_{ik} = \begin{cases} 1 & \text{if } (i, k) \in E \text{ (directed graph) or} \\ & \{i, k\} \in E \text{ (undirected graph)} \\ 0 & \text{otherwise} \end{cases}$$

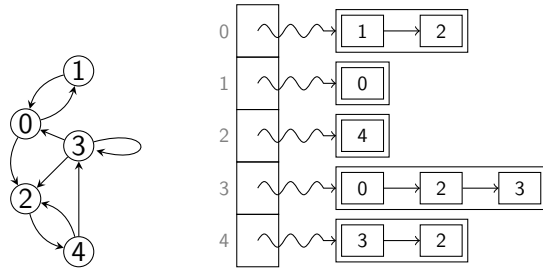


$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

For undirected graphs symmetric

## Adjacency-list Representation

Store for every vertex the list of successors / neighbours.



## Representation: Complexity

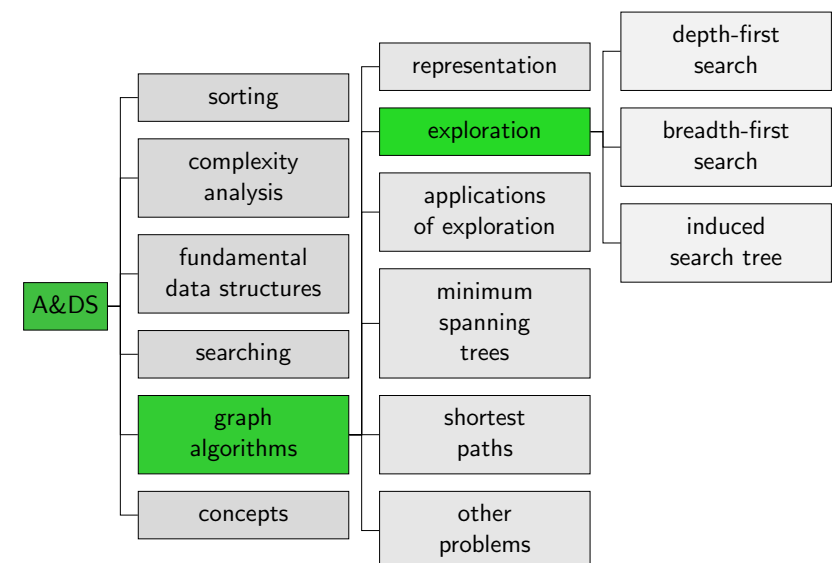
|                             | Adj. matrix | Adj. list                      |
|-----------------------------|-------------|--------------------------------|
| Space                       | $ V ^2$     | $ E  +  V $                    |
| Add edge                    | 1           | 1                              |
| Edge between $u$ and $v$ ?  | 1           | $(\text{out})\text{degree}(v)$ |
| Iterate over outgoing edges | $ V $       | $(\text{out})\text{degree}(v)$ |

Often **sparse** graphs (low average degree)

Which representation?

## C1.4 Graph Exploration

## Content of the Course



## Graph Exploration

- ▶ **Task:** Given a vertex  $v$ , visit all vertices that are reachable from  $v$ .
- ▶ Often used as ingredient of other graph algorithms.
- ▶ **Depth-first search:** go “deep” into the graph (away from  $v$ )
- ▶ **Breadth-first search:** first all neighbours, then neighbours of neighbours, ...

## Depth-first Search

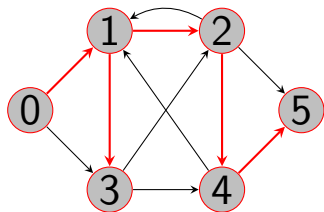
Mark visited vertices

- ▶ Mark  $v$
- ▶ Iterate over the successors/neighbours  $w$  of  $v$ .
  - ▶ If  $w$  not marked, start recursively from  $w$ .

Abbreviation: **DFS**

## Depth-first Search: Example

Here: Visit successors in increasing order of their number.



Depth-first search from start vertex 0 marks vertices in order  
0 - 1 - 2 - 4 - 5 - 3

## Depth-first Search: Algorithm (Recursive)

---

```

1 def depth_first_exploration(graph, node, visited=None):
2     if visited is None:
3         visited = set()
4     if node in visited:
5         return
6     visited.add(node)
7     for s in graph.successors(node):
8         depth_first_exploration(graph, s, visited)

```

---

If we expect that most vertices will be visited:  
bool array instead of set for visited

## Depth-first Vertex Orders

- ▶ **Preorder:** Vertex is included before its children are considered.
- ▶ **Postorder:** Vertex is included when the (recursive) depth-first search of all its children has finished.
- ▶ **Reverse Postorder:** Like postorder, but in reverse order.

```

1 def depth_first_exploration(graph, node, visited):
2     if node in visited:
3         return
4     preorder.append(node)
5     visited.add(node)
6     for s in graph.successors(node):
7         depth_first_exploration(graph, s, visited)
8     postorder.append(node)
9     reverse_postorder.appendleft(node)

```

(Representation of vertex sequence as a deque.)

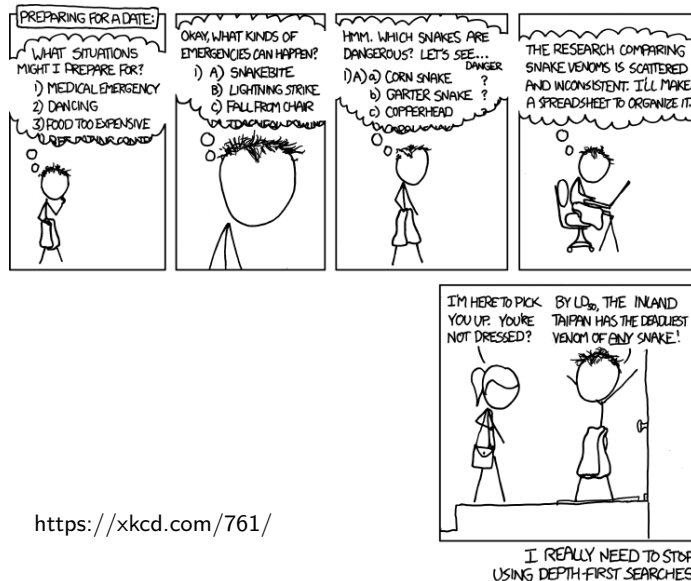
## Depth-first Search: Algorithm (Iterative)

```

1 def depth_first_exploration(graph, node):
2     visited = set()
3     stack = deque()
4     stack.append(node)
5     while stack:
6         v = stack.pop() # LIFO
7         if v not in visited:
8             visited.add(v)
9             for s in graph.successors(v):
10                stack.append(s)

```

## Depth-first Search in Practice



<https://xkcd.com/761/>

## Breadth-first Search

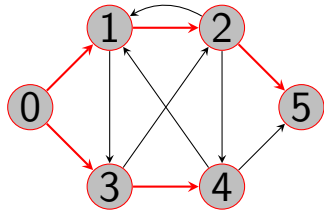
First all neighbours, then neighbours of neighbours, ...

- ▶ Mark  $v$   
→ Distance 0
- ▶ Mark all unmarked successors/neighbours of  $v$   
→ Distance 1
- ▶ Mark all unmarked successors/neighbours of vertices with distance 1.
- ▶ Mark all unmarked successors/neighbours of vertices with distance 2.
- ▶ ...
- ▶ Until vertices of distance  $i$  do not have unmarked successors/neighbours.

Abbreviation: **BFS**

## Breadth-first Search: Example

Here: Visit successors in increasing order of their number.



Breadth-first search from start vertex 0 marks vertices in order 0 - 1 - 3 - 2 - 4 - 5

## Breadth-first Search: Algorithm (Conceptually)

Only difference to iterative depth-first search:

**First-in-first-out** treatment of vertices (instead of last-in-first-out)

---

```

1 def breadth_first_exploration(graph, node):
2     visited = set()
3     queue = deque()
4     queue.append(node)
5     while queue:
6         v = queue.popleft() # FIFO
7         if v not in visited:
8             visited.add(v)
9             for s in graph.successors(v):
10                queue.append(s)

```

---

## Breadth-first Search: Algorithm (Somewhat more Efficient)

We only further consider a vertex when we first run across it.

We can directly mark it as visited and disregard it if we see it again.

---

```

1 def breadth_first_exploration(graph, node):
2     visited = set()
3     queue = deque()
4     visited.add(node)
5     queue.append(node)
6     while queue:
7         v = queue.popleft()
8         for s in graph.successors(v):
9             if s not in visited:
10                visited.add(s)
11                queue.append(s)

```

---

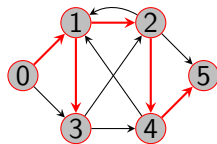
## Running Time

For all algorithm variants:

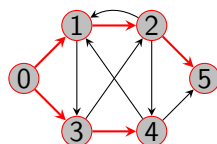
- ▶ Every reachable vertex gets marked.
- ▶ We follow every reachable edge exactly once.
- ▶ Running time  $O(|V| + |E|)$ 
  - ▶ We can restrict this to the reachable vertices and edges.

## Induced Search Tree

The **induced search tree** of a graph exploration contains for every visited vertex (except for the start vertex) an edge from its predecessor in the exploration.



depth-first search



breadth-first search

(induced search tree  $\neq$  binary search tree)

## Induced Search Tree: Example BFS

- ▶ Every vertex has at most one predecessor in the tree.
- ▶ Represent induced search tree by the predecessor relation.
- ▶ The visited vertices are exactly those for which there is a predecessor set: We do not need visited anymore.

```

1 def bfs_with_predecessors(graph, node):
2     predecessor = [None] * graph.no_nodes()
3     queue = deque()
4     # use self-loop for start node
5     predecessor[node] = node
6     queue.append(node)
7     while queue:
8         v = queue.popleft()
9         for s in graph.successors(v):
10            if predecessor[s] is None:
11                predecessor[s] = v
12                queue.append(s)

```

## C1.5 Summary

- ▶ Graphs consist of **vertices** and **edges**.
- ▶ Edges can be **directed** or **undirected**.
- ▶ **Graph exploration** systematically visits all vertices that can be reached from the given vertex.
  - ▶ **Depth-first search** goes “deeper” into the graph whenever possible.
  - ▶ **Breadth-first search** first visits the vertices that are closer to the start vertex.