

Algorithms and Data Structures

B1. Arrays and Linked Lists

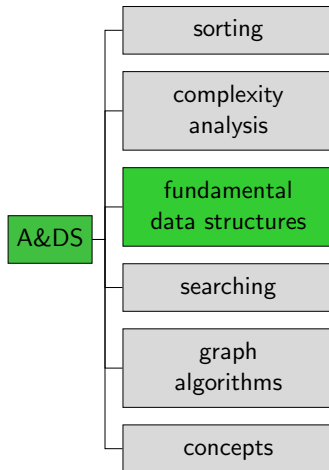
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March 25, 2026

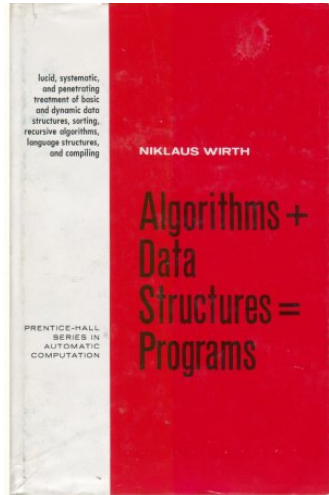
Data Structures

Content of the Course

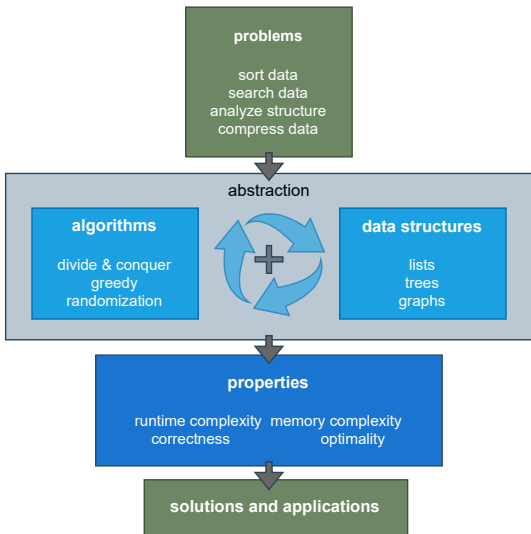


Data Structures

- Programming goes beyond writing algorithms.
 - Organisation of data is central.
- Elegant data structures lead to elegant code.
- Programmers. . .
 - need a catalogue of data structures, and
 - need to know their characteristics.



Overview



Data Structures

Bad programmers worry about the code. Good programmers worry about data structures and their relationships.

Linus Torwalds

Data Structures

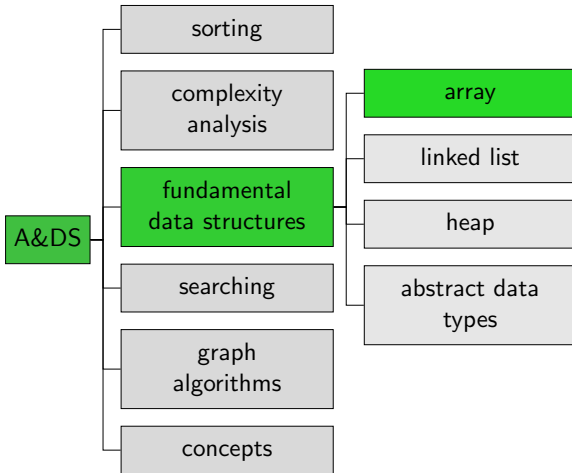
Show me your algorithm and conceal your data structures,
and I shall continue to be mystified.

Show me your data structures, and I won't usually need your
algorithm; it will be obvious.

Fred Brooks (paraphrased)

Arrays

Content of the Course



Data Structure: Array

- **Arrays** are one of the fundamental data structures, that can be found in (almost) every programming language.
- An array stores a sequence of elements (of the same memory size) as a contiguous sequence of bytes in memory.
- The number of elements is fixed.
- We can access elements by their index.

In Java:

```
byte [] myByteArray = new byte [100];  
char [] myCharArray = new char [50];
```

Example: char Array

- One char occupies 1 byte.
- The first element is at memory address 2000 (7D0 in hexadecimal).
- The first element has index 0.
- Then the element with index i is at address $2000 + i$.

Memory address	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
(hex)	0x7D0	0x7D1	0x7D2	0x7D3	0x7D4	0x7D5	0x7D6	0x7D7	0x7D8	0x7D9	0x7DA
	h	e	l	l	o	_	w	o	r	l	d
Index	0	1	2	3	4	5	6	7	8	9	10

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
 - the size of the array (number of elements) and
 - the address of the start of the allocated memory.

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 - `get(i)` – return element at position i
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Observation

Complexity is direct consequence of data representation.

Lists in Python

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→ **dynamic array**

Dynamic Arrays

(Static) arrays have fixed capacity that must be specified at allocation.

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Additional operations:

- `append(x)` (or `push`) – append element `x` at the end.
- `insert(i, x)` – insert element `x` at position `i`.
- `pop()` – remove the last element.
- `remove(i)` – remove the element at position `i`.

Changing the Array Size: Naive Method

- append and insert increase the size of the array.
- pop and remove decrease the size.
- Naive method:
 - Allocate new memory range that is one element larger/smaller.
 - Move all (but the potentially removed) element over.

Changing the Array Size: Naive Method

- `append` and `insert` increase the size of the array.
- `pop` and `remove` decrease the size.
- Naive method:
 - Allocate new memory range that is one element larger/smaller.
 - Move all (but the potentially removed) element over.

With this approach, these operations would take linear time in the current size of the array!

Better Approach: Overallocate Memory

- Allocate more memory than needed for the current array size.
- Distinguish
 - **capacity** = number of elements that fit in the allocated space.
 - **size** = number of currently contained elements.

Better Approach: Append/Insert

Append

- If capacity $>$ size:
 - Write the new element to position size and increment size.
- Otherwise (capacity = size):
 - Allocate new memory that is larger than necessary (e.g. twice the previous capacity).
 - Copy all elements to the new memory (release the old one).
 - Update the capacity and continue as in case capacity $>$ size.

Insert at pos i : Analogously but move all elements at positions i to size-1 one position to the right before writing the new element to i .

Better Approach: Pop/Remove

- If capacity much too large (e.g. $\text{capacity} > 4 \cdot \text{size}$), move all elements into new smaller memory range (e.g. with half the previous capacity)
- Pop: remove element at position $\text{size} - 1$ and decrement size.
- Remove: remove element at position i and move all elements right of i one position to the left, decrement size.

Amortized Analysis

- Worst-case analysis often pessimistic: append takes linear time if new memory allocated but in a sequence of append operations, this will happen rarely.
- **Amortized analysis** determines the average cost of an operation over an entire sequence of operations.
- Don't confuse this with an average-case analysis.
- Different methods
 - Aggregate analysis
 - Accounting method ← now
 - Potential method

Accounting Method

- Assign charges to operations.
- Some operations charged more or less than they actually cost.
- If charged more: save difference as credit
- If charged less: use up some credit to pay for the difference.
- Credit must be non-negative all the time.
- Then the total amortized cost is always an upper bound on the actual total costs so far.

Accounting Method: Append I

- Append without resize: constant cost (e.g. 1).
Just insert the element at the right position.
- Append with resize: linear cost (1 for every element).
 - If the append element gets position 2^i ($i \in \mathbb{N}_{>0}$),
 - we first allocate overall space for 2^{i+1} elements, and
 - move all $2^i - 1$ existing elements to the new space.

Accounting Method: Append II

Execute a sequence of append operations starting on empty array.

size (after append)	capacity	cost
1	2	1
2	2	1
3	4	3
4	4	1
5	8	5
6	8	1
7	8	1
8	8	1
9	16	9
10	16	1

Accounting Method: Append II

Execute a sequence of append operations starting on empty array.
Charge cost 3 for every append operation.

size (after append)	capacity	cost	charge	credit
1	2	1	3	2
2	2	1	3	4
3	4	3	3	4
4	4	1	3	6
5	8	5	3	4
6	8	1	3	6
7	8	1	3	8
8	8	1	3	10
9	16	9	3	4
10	16	1	3	6

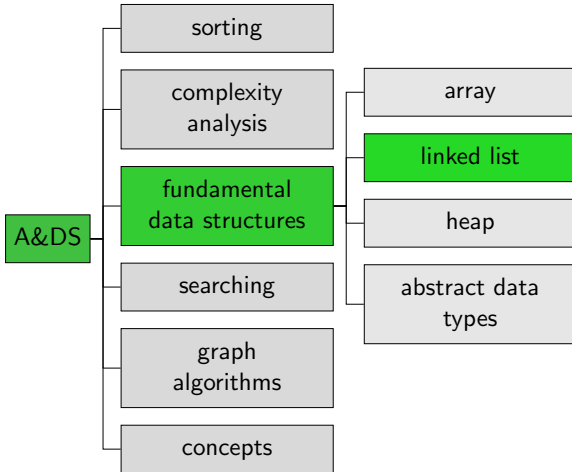
Charging 3 per operation covers all “running time costs”.
→ Append has constant amortized running time.

Worst-Case Running Time Array

Operation	Array
Access element by position	$O(1)$
Prepend/remove first element	$O(n)$
Append	$O(1)$ (amortized)
Remove last element	$O(1)$ (amortized)
Insert, remove from the middle	$O(n)$
Traverse all elements	$O(n)$

Linked Lists

Content of the Course



Motivation

- Arrays need a large continuous block of memory.
- Inserting elements at arbitrary positions is expensive.

Alternative that allows us to distribute the elements in memory?

Question?

- How can we order elements that are distributed in memory?

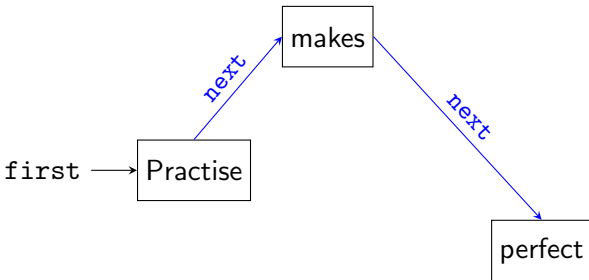
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Practise

perfect

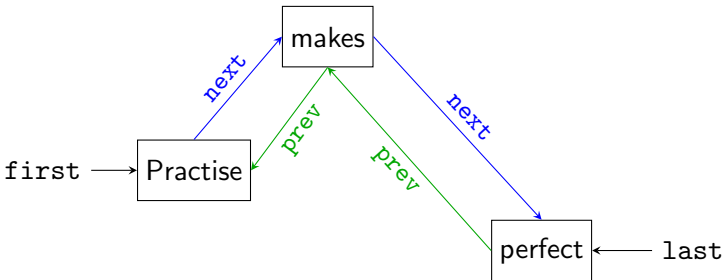
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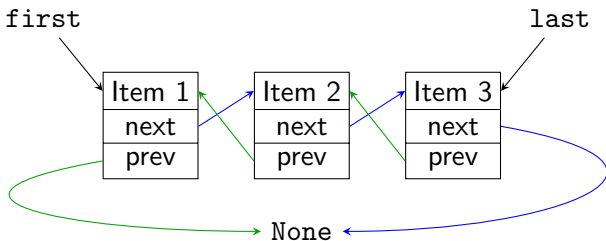
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(Doubly) Linked Lists

- Every node stores its entry as well as a reference/pointer next to its successor and a reference/pointer prev to its predecessor.
- Need special value for next of the last element and prev of the first element (e.g. None).
- The list maintains a pointer to the first and the last node.



Jupyter Notebook



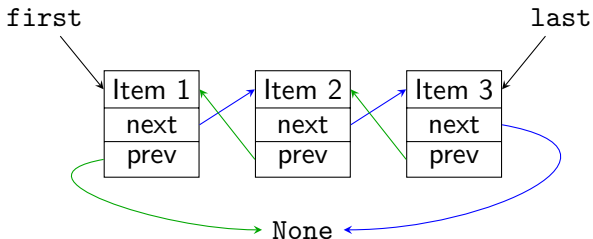
Jupyter notebook: `doubly_linked_lists.ipynb`

Doubly Linked Lists: Implementation

```
1  class Node:
2      def __init__(self, item, next=None, prev=None):
3          self.item = item
4          self.next = next
5          self.prev = prev
6
7  class DoublyLinkedList:
8      def __init__(self):
9          self.first = None
10         self.last = None
11
12         def is_empty(self):
13             return self.first is None
14
15         # other methods on next slides
```

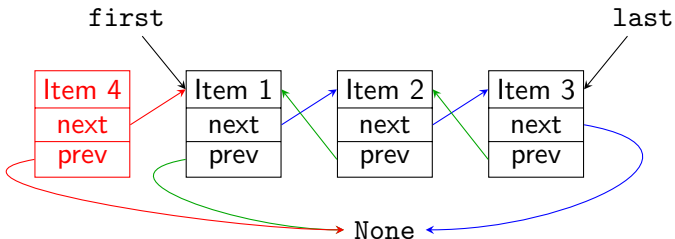
Doubly Linked Lists: prepend

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15     def prepend(self, item):
16         if self.is_empty():
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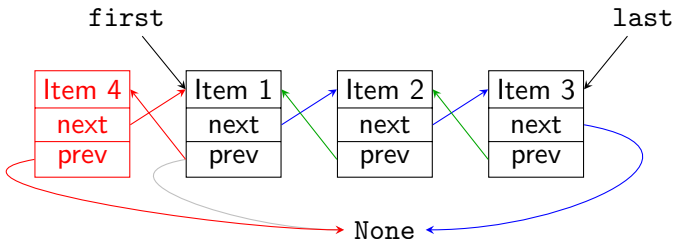
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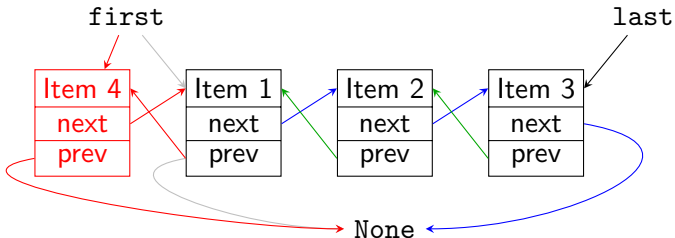
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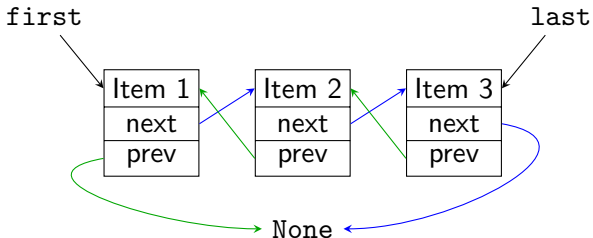
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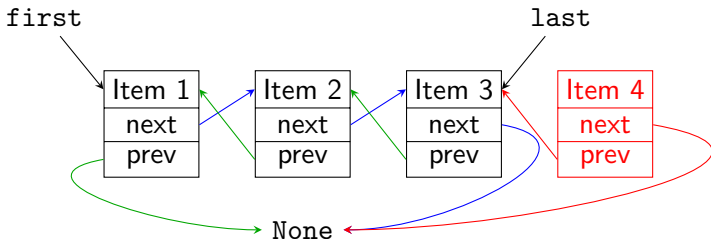
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30             self.last.next = node
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```



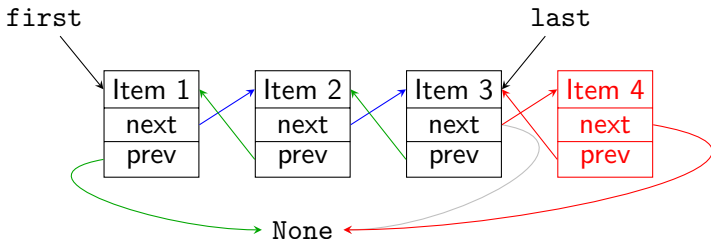
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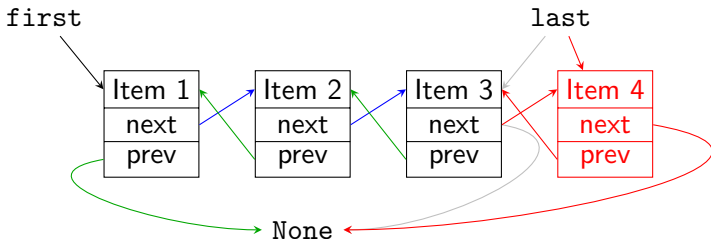
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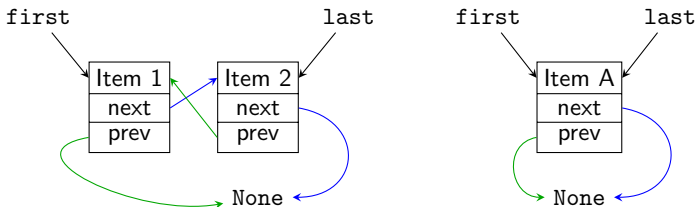
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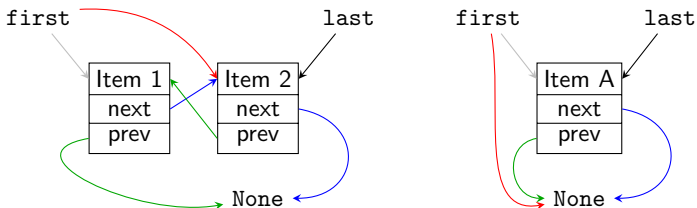
Doubly Linked Lists: remove_first

```
33     def remove_first(self):  
34         if self.is_empty():  
35             raise Exception("removing from empty list")  
36         item = self.first.item  
37         self.first = self.first.next  
38         if self.first is not None:  
39             self.first.prev = None  
40         else:  
41             self.last = None  
42         return item
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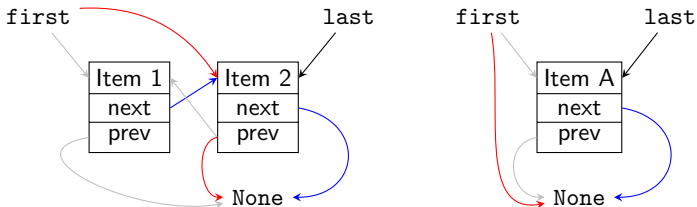
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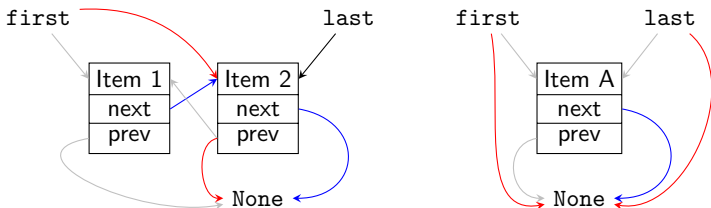
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```



Doubly Linked Lists: remove_last

Removing the last element is analogous to removing the first one:

```
44     def remove_last(self):
45         if self.is_empty():
46             raise Exception("removing from empty list")
47         item = self.last.item
48         self.last = self.last.prev
49         if self.last is not None:
50             self.last.next = None
51         else:
52             self.first = None
53         return item
```

Worst-Case Running Time Array / Doubly Linked List

Operation	Array	Doubly Linked List
Prepend/remove first element	$O(n)$	$O(1)$
Append	$O(1)$ (amort.)	$O(1)$
Remove last element	$O(1)$ (amort.)	$O(1)$
Insert, remove in the middle	$O(n)$	$O(n)/O(1)^*$
Traverse all elements	$O(n)$	$O(n)$
Find an element	$O(n)$	$O(n)$
Access element by position	$O(1)$	–
Additional memory	$O(1)$	$O(n)$

* constant, if node at the position is parameter

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Take-home Message

Different data structures have different trade-offs.

Summary

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- An **amortized analysis** determines the average cost of an operation over an entire sequence of operations.
- Arrays and linked lists store sequences of items.
 - **Arrays** store items in a continuous space and can efficiently access an item by index.
 - **(Doubly) linked lists** store items in nodes with references to the next and to the previous node.