

Algorithms and Data Structures

A5. Runtime Analysis: Introduction and Selection Sort

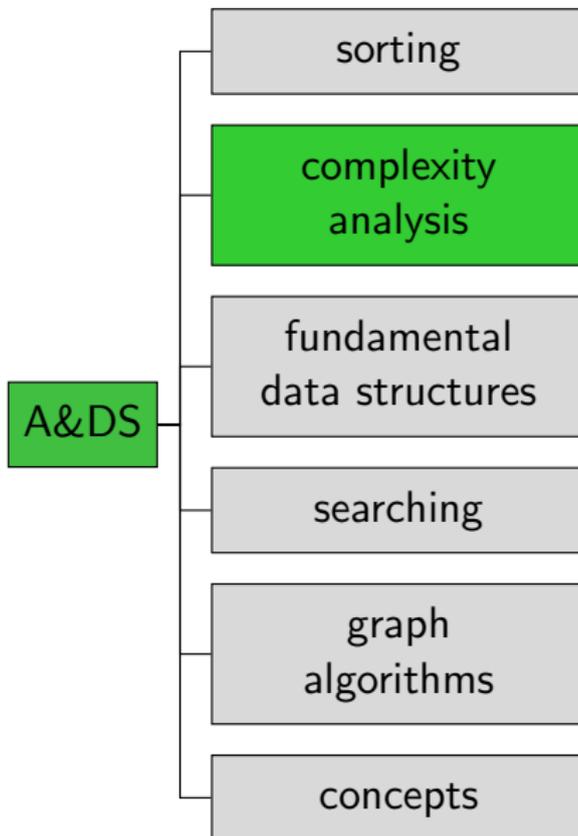
Gabriele Röger and Patrick Schneider

University of Basel

March 4, 2026

Runtime Analysis in General

Content of the Course



Exact Runtime Analysis Unrealistic

- **Would be nice:** formula that determines for a specific input how long the computation will take.
- **Exact runtime prediction is hard** because of too many influencing factors.
 - Speed and architecture of the computer
 - Programming language
 - Compiler version
 - Current load (what else is running?)
 - Caching behavior

We neither can nor want to consider all this in a formula.

Runtime Analysis: 1st Simplification

Don't measure time but count operations

What is an operation?

- Ideally: one line of machine code or – even more precisely – one processor cycle
- Instead: constant-time operations
 - Constant time: running time independent of input.
 - Ignore runtime differences of different operations.
 - E.g. addition, assignments, branching, function call.
 - **Roughly**: operation = one line of code.
 - **But**: also consider what's behind it
e.g. steps inside the called function.

Running time roughly proportional to the number of operations

Runtime Analysis: 2nd Simplification

Don't count exactly but use bounds!

- Mostly considering upper bounds
How many steps does it take at most?
- Sometimes also lower bound
How many steps are at least executed?

''running time'' for bound on number of executed operations

Runtime Analysis: 3rd Simplification

Bounds only relative to the input size

- $T(n)$: running time for input of size n
- For adaptive algorithms we distinguish
 - Best case
running time for best possible input of size n
 - Worst case
running time for worst possible input of size n
 - Average case
average running time over all inputs of size n

Cost Models

Sometimes: analysis wrt. **cost model**

- Identify fundamental operations for the algorithm class e.g. for sorting algorithms.
 - Key comparison
 - Swap of two elements or movement of an element
- Analyze number of these operations.

Example from C++ Reference

function template

std::sort

<algorithm>

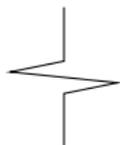
```
default (1)  template <class RandomAccessIterator>
              void sort (RandomAccessIterator first, RandomAccessIterator last);
custom (2)   template <class RandomAccessIterator, class Compare>
              void sort (RandomAccessIterator first, RandomAccessIterator last, Compare comp);
```

Sort elements in range

Sorts the elements in the range `[first, last)` into ascending order.

The elements are compared using `operator<` for the first version, and `comp` for the second.

Equivalent elements are not guaranteed to keep their original relative order (see [stable_sort](#)).

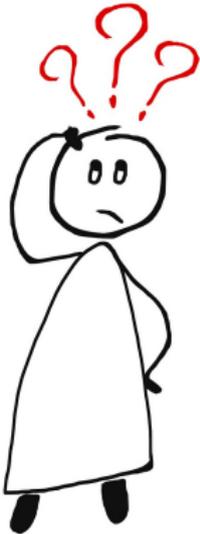


Complexity

On average, linearithmic in the [distance](#) between `first` and `last`: Performs approximately $N \cdot \log_2(N)$ (where N is this distance) comparisons of elements, and up to that many element swaps (or moves).

<http://www.cplusplus.com/reference/algorithm/sort/>

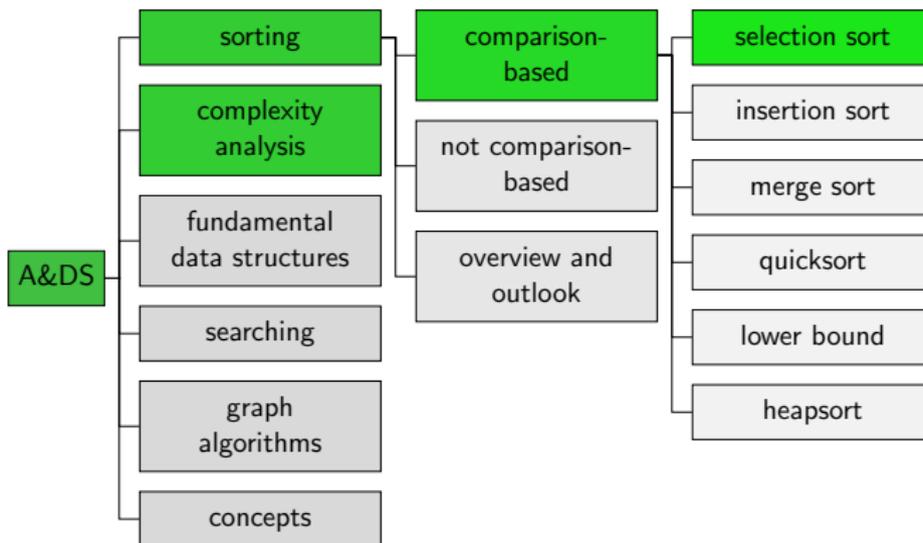
Questions



Questions?

Example: Selection Sort

Content of the Course



Selection Sort: Algorithm

```
1 def selection_sort(array):
2     n = len(array)
3     for i in range(n - 1): # i = 0, ..., n-2
4         # find index of minimum element at positions i, ..., n-1
5         min_index = i
6         for j in range(i + 1, n): # j = i+1, ..., n-1
7             if array[j] < array[min_index]:
8                 min_index = j
9         # swap element at position i with minimum element
10        array[i], array[min_index] = array[min_index], array[i]
```

Selection Sort with Cost Model

```
1 def selection_sort(array):
2     n = len(array)
3     for i in range(n - 1): # i = 0, ..., n-2
4         # find index of minimum element at positions i, ..., n-1
5         min_index = i
6         for j in range(i + 1, n): # j = i+1, ..., n-1
7             if array[j] < array[min_index]:
8                 min_index = j
9         # swap element at position i with minimum element
10        array[i], array[min_index] = array[min_index], array[i]
```

On an input of size n , how often does the algorithm swap two elements?



Selection Sort with Cost Model

```
1 def selection_sort(array):
2     n = len(array)
3     for i in range(n - 1): # i = 0, ..., n-2
4         # find index of minimum element at positions i, ..., n-1
5         min_index = i
6         for j in range(i + 1, n): # j = i+1, ..., n-1
7             if array[j] < array[min_index]:
8                 min_index = j
9         # swap element at position i with minimum element
10        array[i], array[min_index] = array[min_index], array[i]
```

→ $n-1$ swaps of two elements (“linear”)

Selection Sort with Cost Model

```
1 def selection_sort(array):
2     n = len(array)
3     for i in range(n - 1): # i = 0, ..., n-2
4         # find index of minimum element at positions i, ..., n-1
5         min_index = i
6         for j in range(i + 1, n): # j = i+1, ..., n-1
7             if array[j] < array[min_index]:
8                 min_index = j
9         # swap element at position i with minimum element
10        array[i], array[min_index] = array[min_index], array[i]
```

→ $n-1$ swaps of two elements (“linear”)

→ $\binom{n}{2} = \frac{1}{2}n(n-1)$ key comparisons (“quadratic”)

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:
 - Constant a for no. of operations in lines 7 and 8
 - Constant b for no. of operations in lines 5 and 10

i | # operations

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:
 - Constant a for no. of operations in lines 7 and 8
 - Constant b for no. of operations in lines 5 and 10

i	# operations
0	$a(n-1) + b$
1	$a(n-2) + b$
	...

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:
 - Constant a for no. of operations in lines 7 and 8
 - Constant b for no. of operations in lines 5 and 10

i	# operations
0	$a(n-1) + b$
1	$a(n-2) + b$
	...
$n-2$	$a \cdot 1 + b$

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:
 - Constant a for no. of operations in lines 7 and 8
 - Constant b for no. of operations in lines 5 and 10

i	# operations
0	$a(n-1) + b$
1	$a(n-2) + b$
	...
$n-2$	$a \cdot 1 + b$

- Total: $T(n) = \sum_{i=0}^{n-2} (a(n - (i + 1)) + b)$

Selection Sort: Analysis II

$$T(n) = \sum_{i=0}^{n-2} (a(n - (i + 1)) + b)$$

Selection Sort: Analysis II

$$\begin{aligned} T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\ &= \sum_{i=1}^{n-1} (a(n - i) + b) \end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1)\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1)\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1)\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1)n\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1)n \\&\leq \frac{1}{2}an^2 + bn^2\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1)n \\&\leq \frac{1}{2}an^2 + bn^2 &= \left(\frac{1}{2}a + b\right)n^2\end{aligned}$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2} (a(n - (i + 1)) + b) \\&= \sum_{i=1}^{n-1} (a(n - i) + b) \\&= a \sum_{i=1}^{n-1} (n - i) + b(n - 1) \\&= \frac{1}{2}a(n - 1)n + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1) \\&\leq \frac{1}{2}an^2 + b(n - 1)n \\&\leq \frac{1}{2}an^2 + bn^2 &= \left(\frac{1}{2}a + b\right)n^2\end{aligned}$$

\Rightarrow with $c' = \left(\frac{1}{2}a + b\right)$ it holds for $n \geq 1$ that $T(n) \leq c' \cdot n^2$

Selection Sort: Analysis III

Too generous bound?

We show for $n \geq 2$: $T(n) \geq c \cdot n^2$ for some constant c

Selection Sort: Analysis III

Too generous bound?

We show for $n \geq 2$: $T(n) \geq c \cdot n^2$ for some constant c

$$\begin{aligned} T(n) &= \dots = \frac{1}{2}a(n-1)n + b(n-1) \\ &\geq \frac{1}{2}a(n-1)n \\ &\geq \frac{1}{4}an^2 \quad \left(n-1 \geq \frac{1}{2}n \text{ for } n \geq 2 \right) \end{aligned}$$

\Rightarrow with $c = \frac{1}{4}a$ it holds for $n \geq 2$ that $T(n) \geq c \cdot n^2$

Selection Sort: Analysis III

Too generous bound?

We show for $n \geq 2$: $T(n) \geq c \cdot n^2$ for some constant c

$$\begin{aligned} T(n) &= \dots = \frac{1}{2}a(n-1)n + b(n-1) \\ &\geq \frac{1}{2}a(n-1)n \\ &\geq \frac{1}{4}an^2 \quad (n-1 \geq \frac{1}{2}n \text{ for } n \geq 2) \end{aligned}$$

\Rightarrow with $c = \frac{1}{4}a$ it holds for $n \geq 2$ that $T(n) \geq c \cdot n^2$

Theorem

Selection sort has **quadratic running time**, i.e., there are constants $c > 0$, $c' > 0$, $n_0 > 0$ such that for $n \geq n_0$: $cn^2 \leq T(n) \leq c'n^2$.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.
- With 10 thousand elements, we wait
 $10^{-8} \cdot (10^4)^2 = 1$ second.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.
- With 10 thousand elements, we wait
 $10^{-8} \cdot (10^4)^2 = 1$ second.
- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.
- With 10 thousand elements, we wait
 $10^{-8} \cdot (10^4)^2 = 1$ second.
- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.
- With 1 million elements $10^{-8} \cdot (10^6)^2$ seconds = 2.77 hours.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.
- With 10 thousand elements, we wait
 $10^{-8} \cdot (10^4)^2 = 1$ second.
- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.
- With 1 million elements $10^{-8} \cdot (10^6)^2$ seconds = 2.77 hours.
- With 1 billion elements $10^{-8} \cdot (10^9)^2$ seconds = 317 years.
1 billion numbers with 4 bytes/number are "‘only’" 4 GB.

Selection Sort: Analysis IV

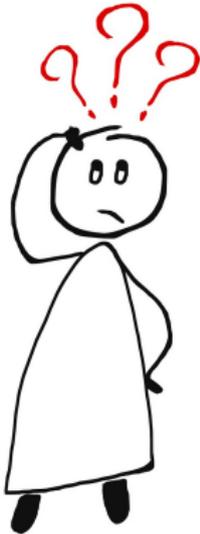
Quadratic running time: twice as large input, fourfold running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average 10^{-8} sec.
- With 1000 elements, we wait
 $10^{-8} \cdot (10^3)^2 = 10^{-8} \cdot 10^6 = 10^{-2} = 0.02$ seconds.
- With 10 thousand elements, we wait
 $10^{-8} \cdot (10^4)^2 = 1$ second.
- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.
- With 1 million elements $10^{-8} \cdot (10^6)^2$ seconds = 2.77 hours.
- With 1 billion elements $10^{-8} \cdot (10^9)^2$ seconds = 317 years.
1 billion numbers with 4 bytes/number are "only" 4 GB.

Quadratic running time problematic for large inputs

Questions



Questions?

Summary

Summary

- Runtime analysis considers **bounds** on the **number of executed operations**.
 - We don't count exactly.
 - We ignore how long each operation actually takes.
 - Running time should be roughly proportional to the number of operations.
- **Selection sort** has **quadratic running time** and performs a linear number of swaps and a quadratic number of key comparisons.