Algorithms and Data Structures

A3. Sorting I: Selection and Insertion Sort

Gabriele Röger and Patrick Schnider

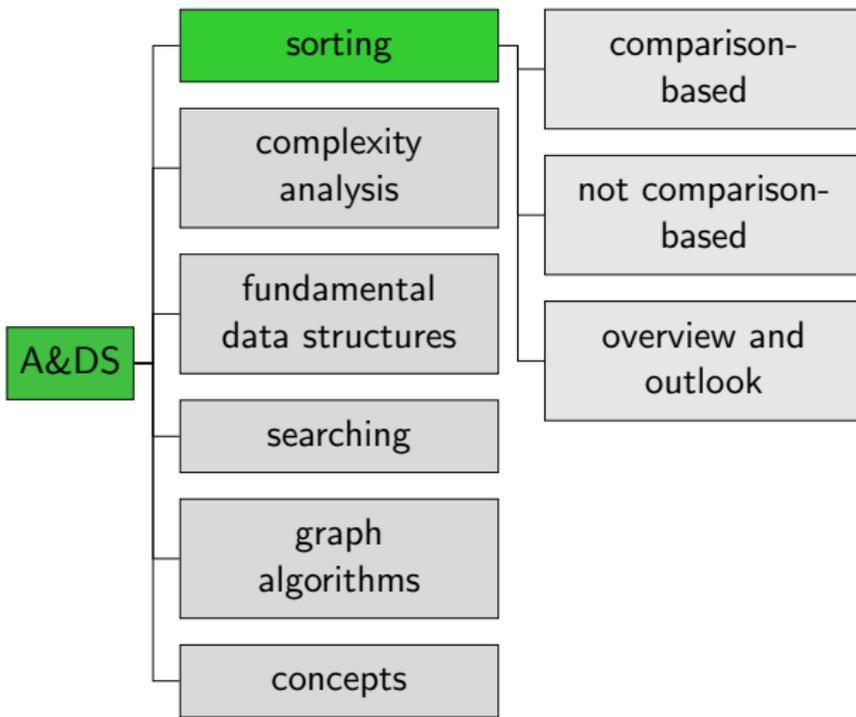University of Basel

February 19, 2026

A3.1 Sorting

A3.2 Selection Sort

A3.3 Insertion Sort

A3.4 Summary

A3.1 Sorting

Content of the Course

Relevance

sorting data important for many applications, such as

- sorted presentation (e.g. on website)
  - products sorted by price, rating, . . .
  - account transactions sorted by transaction date
- preprocessing for many efficient search algorithms
  - How quickly can you find a number in a (physical) telephone book? How quickly could you do so if the entries were not sorted?
- subroutine of many other algorithms
  - e.g. a program that renders layered graphical objects might sort them to determine where objects are covered by other objects

Journal "Computing in Science & Engineering" lists Quicksort as one of the 10 most important algorithms of the 20th century.

Sorting Problem

Sorting Problem

Input

▶ sequence of n elements $e_1, \ldots, e_n$

▶ each element e_i has key $k_i = key(e_i)$

▶ partial order $\leq$ on the keys
  reflexive: $k \leq k$
  transitive: $k \leq k'$ and $k' \leq k'' \Rightarrow k \leq k''$
  antisymmetric: $k \leq k'$ and $k' \leq k \Rightarrow k = k'$

Output

▶ Sequence of the same elements sorted
  according to the ordering relation on its keys

Notation: also $e \leq e'$ for $key(e) \leq key(e')$

Sorting Problem: Examples

Example
Input: $\langle 3, 6, 2, 3, 1 \rangle$, $key(e) = e$, $\leq$ on the integers
Output: $\langle 1, 2, 3, 3, 6 \rangle$

Example
Input: list of all students of the Univ. of Basel,
        $key(e) = \langle$place of residence of $e\rangle$, lexicographic order
Output: list of all students, sorted by their place of residence
Is the output uniquely defined?

In this course: mostly integers, $key(e) = e$ and $\leq$ on integers
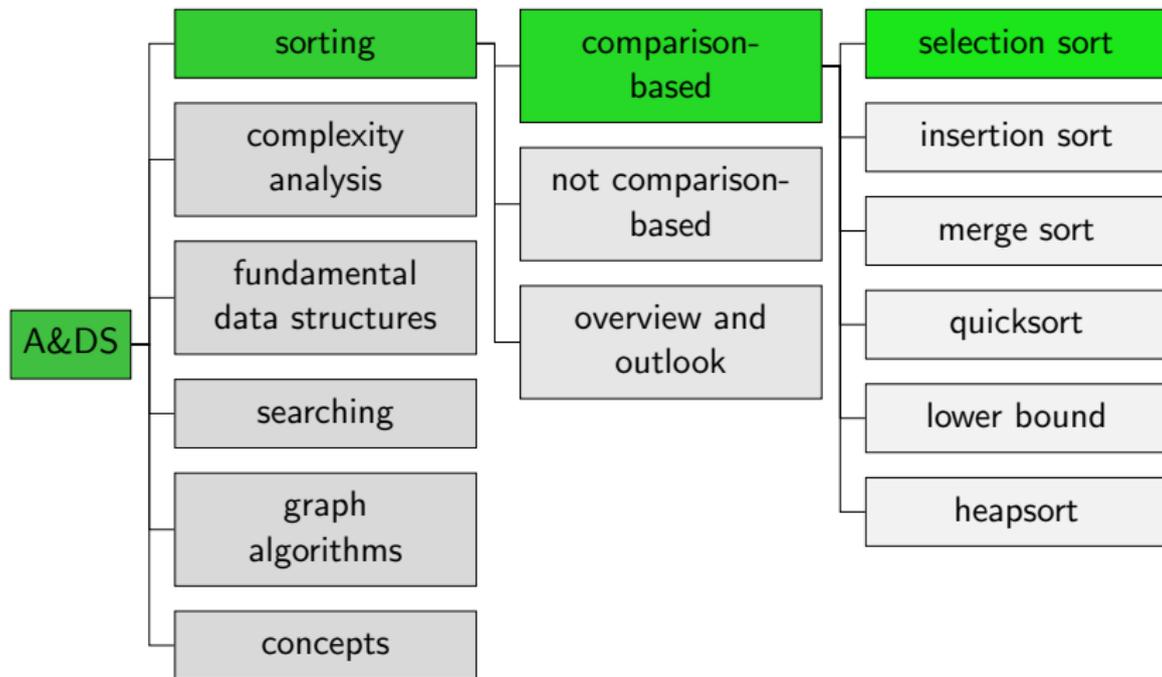
Interesting Properties of Sorting Algorithms

- ▶ running time: how many key comparisons and swaps of elements are executed?
    - ▶ adaptive: algorithms faster if input already (partially) sorted
- ▶ space consumption: how much space is used on top of the space occupied by the input sequence?
    - ▶ explicitly or in call stack
    - ▶ in-place: needs no additional storage beyond the input array and a constant amount of space (independent of input size)
- ▶ stable: elements with the same value appear in the output sequence in the same order as they do in the input sequence
- ▶ comparison-based: uses only key comparisons and swaps of elements
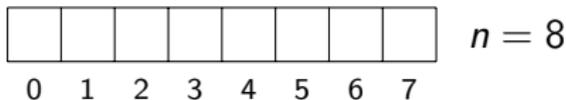
Content of the Course
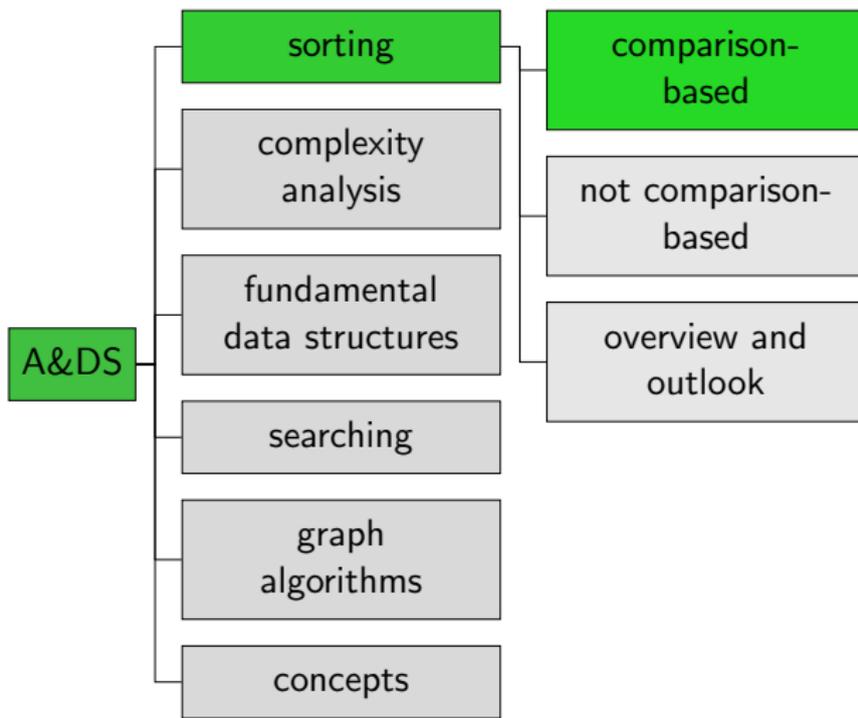
A3.2 Selection Sort

Content of the Course

```
                    ┌──────────────┐      ┌──────────────┐      ┌──────────────┐
                    │   sorting    │──────│ comparison-  │──────│selection sort│
                    └──────────────┘      │    based     │      └──────────────┘
                    ┌──────────────┐      └──────────────┘      ┌──────────────┐
                    │  complexity  │                            │insertion sort│
                    │   analysis   │      ┌──────────────┐      └──────────────┘
                    └──────────────┘      │not comparison-│     ┌──────────────┐
                    ┌──────────────┐      │    based     │      │  merge sort  │
                    │  fundamental │      └──────────────┘      └──────────────┘
          ┌─────┐   │data structures│     ┌──────────────┐      ┌──────────────┐
          │A&DS │───└──────────────┘      │ overview and │      │  quicksort   │
          └─────┘   ┌──────────────┐      │   outlook    │      └──────────────┘
                    │  searching   │      └──────────────┘      ┌──────────────┐
                    └──────────────┘                            │ lower bound  │
                    ┌──────────────┐                            └──────────────┘
                    │    graph     │                            ┌──────────────┐
                    │  algorithms  │                            │   heapsort   │
                    └──────────────┘                            └──────────────┘
                    ┌──────────────┐
                    │   concepts   │
                    └──────────────┘
```

Selection Sort: Informally

$n = 8$
0 1 2 3 4 5 6 7

▶ identify smallest element at positions $0, \ldots, n-1$
  and swap it to position 0
▶ identify smallest element at positions $1, \ldots, n-1$
  and swap it to position 1
▶ . . .
▶ identify smallest element at positions $n-2, n-1$
  and swap it to position n - 2

Selection Sort: Example

| 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |

| 1 | 7 | 2 | 9 | 7 | 3 | 4 | 5 |

| 1 | 2 | 7 | 9 | 7 | 3 | 4 | 5 |

| 1 | 2 | 3 | 9 | 7 | 7 | 4 | 5 |

| 1 | 2 | 3 | 4 | 7 | 7 | 9 | 5 |

| 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |

| 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |

| 1 | 2 | 3 | 4 | 5 | 7 | 7 | 9 |

Selection Sort: Algorithm

```python
 1 def selection_sort(array):
 2     n = len(array)
 3     for i in range(n - 1):  # i = 0, ..., n-2
 4         # find index of minimum element at positions i, ..., n-1
 5         min_index = i
 6         for j in range(i + 1, n):  # j = i+1, ..., n-1
 7             if array[j] < array[min_index]:
 8                 min_index = j
 9         # swap element at position i with minimum element
10         array[i], array[min_index] = array[min_index], array[i]
```

Selection Sort: Example

i	min_ind.	0	1	2	3	4	5	6	7	
		3	7	2	9	7	1	4	5	
0	5	3	7	2	9	7	1	4	5	looking for minimum
1	2	1	7	2	9	7	3	4	5	among dark entries
2	5	1	2	7	9	7	3	4	5	
3	6	1	2	3	9	7	7	4	5	
4	7	1	2	3	4	7	7	9	5	
5	5	1	2	3	4	5	7	9	7	red entry is
6	7	1	2	3	4	5	7	9	7	found minimum
		1	2	3	4	5	7	7	9	

gray entries already sorted

Correctness

Correctness of an algorithm

An algorithm for a computational problem is correct if for every problem instance provided as input, it

- ▶ halts, i.e. it finishes its computation in finite time, and
- ▶ determines a correct solution to the problem instance.

Correctness of Selection Sort

▶ invariant: property that is true during the entire execution of the algorithm

▶ invariant 1: at the end of each iteration of the outer loop, all elements at positions $\leq i$ are sorted.

▶ Invariant 2: at the end of each iteration of the outer loop, all elements at a position $> i$ are larger ($\geq$) than all elements at a position $\leq i$.

▶ correctness of invariants by (joint) induction

▶ after the last iteration, all elements except for the last one are in the correct order and the last one is larger ($\geq$) than the second-last. $\rightarrow$ entire sequence sorted

▶ Termination: $n - 1$ iterations of outer loop, each with fewer than n iterations of inner loop $\rightarrow$ finite runtime

Properties of Selection Sort

▶ in-place: additional storage does not depend on input size
▶ running time: does only depend on the size of the input
  (not adaptive)
  exact analysis: later chapter
▶ not stable: can swap the element at position i behind an
  element with an equal key, which will not be "repaired" later.

Jupyter Notebook

Jupyter notebook: `selection_sort.ipynb`

A3.3 Insertion Sort

Content of the Course
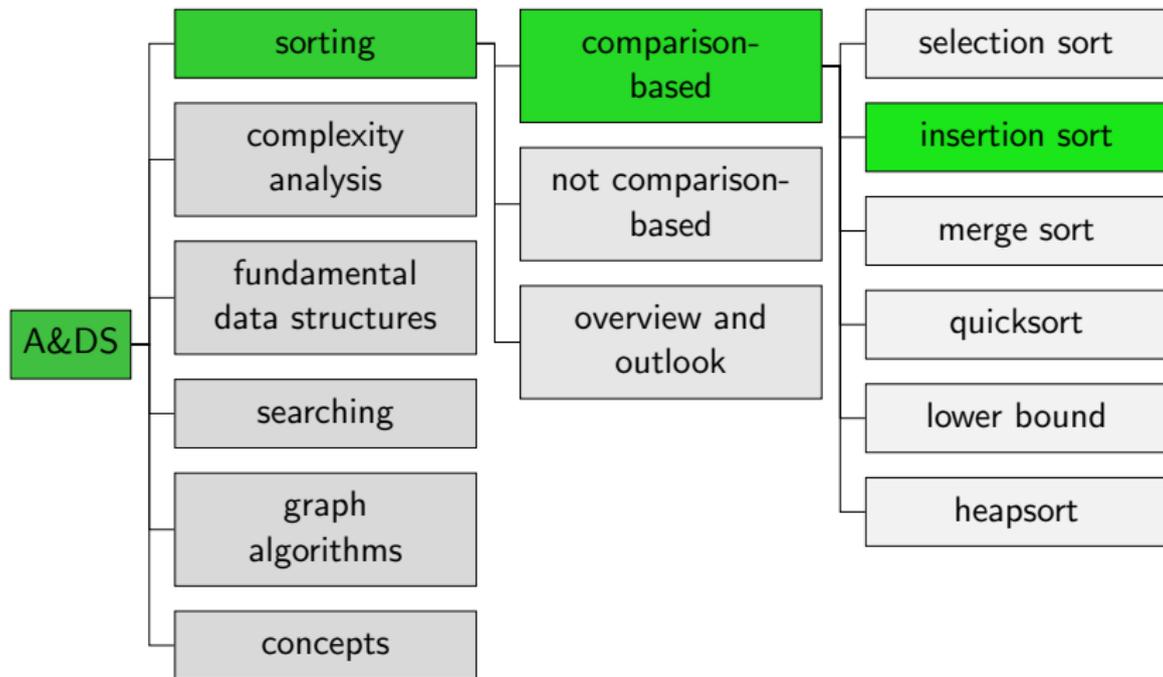
Insertion Sort: Informally

- ▶ similar to common method for sorting a hand of playing cards
- ▶ elements subsequently moved to correct position in the already sorted part of the sequence
- ▶ larger elements correspondingly moved to the right

Insertion Sort: Example

i	0	1	2	3	4	5	6	7
	3	7	2	9	7	1	4	5
1	3	7	2	9	7	1	4	5
2	2	3	7	9	7	1	4	5
3	2	3	7	9	7	1	4	5
4	2	3	7	7	9	1	4	5
5	1	2	3	7	7	9	4	5
6	1	2	3	4	7	7	9	5
7	1	2	3	4	5	7	7	9

gray entries
not moved

red entry moved
into sorted range

black entries moved
one position to the right

Insertion Sort: Algorithm

```
1 def insertion_sort(array):
2     n = len(array)
3     for i in range(1, n):  # i = 1, ..., n - 1
4         # move array[i] to the left until it is
5         # at the correct position.
6         j = i
7         while j > 0 and array[j - 1] > array[j]:
8             # not yet at final position.
9             # swap array[j] and array[j-1]
10            array[j], array[j-1] = array[j-1], array[j]
11            j -= 1
```

Insertion Sort: Algorithm (Slightly Faster)

previous variant: most assignments to array[j-1] unnecessary

```python
1 def insertion_sort(array):
2     for i in range(1, len(array)):
3         val = array[i]
4         j = i
5         while j > 0 and array[j - 1] > val:
6             array[j] = array[j - 1]
7             j -= 1
8         array[j] = val
```

runtime analysis (later): no fundamental difference
nevertheless: preferable if direct assignment possible

Properties of Insertion Sort

- ▶ in-place: additional storage does not depend on input size
- ▶ running time: adaptive for partially sorted inputs
  - ▶ with already sorted input, immediate exit from inner loop
  - ▶ with reversely sorted input, every element moved step-by-step to the front

  exact analysis: later

- ▶ stable: elements only moved to the left as long it is swapped with a strictly larger element.
  $\rightarrow$ cannot change relative order with an equal element

A3.4 Summary

Summary

▶ selection sort and insertion sort are two simple sorting algorithms.
▶ selection sort builds the sorted sequence from left to right by successively swapping a minimal element from the unsorted range to the end of the sorted range.
▶ insertion sort considers the elements from left to right and moves them to the correct position in the already sorted range at the beginning of the sequence.