Theory of Computer Science

D4. Some NP-Complete Problems, Part I

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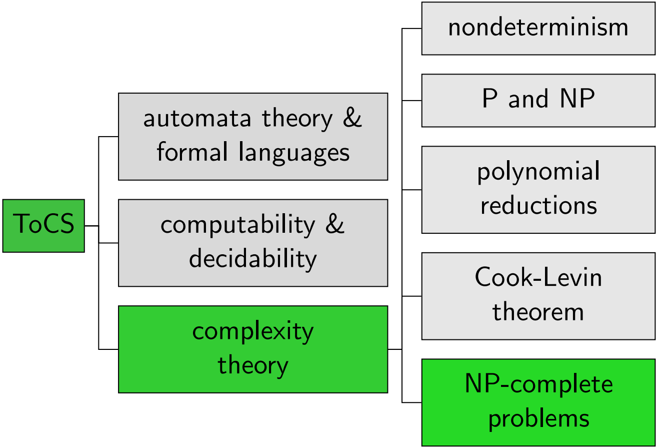
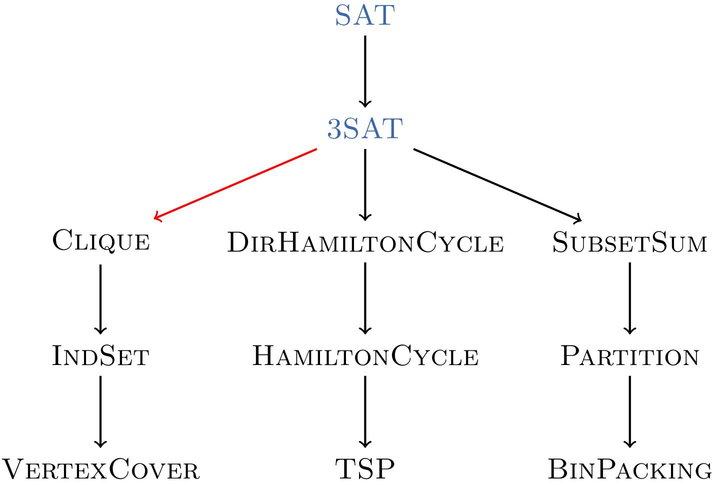
Content of the Course



D4.1 Graph Problems

3SAT $\leq_p$ CLIQUE



Clique

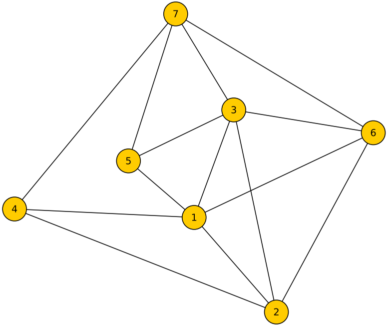
Definition (Clique)

The problem Clique is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a clique of size at least K,
i. e., a set of vertices $C \subseteq V$ with $|C| \geq K$
and $\{u, v\} \in E$ for all $u, v \in C$ with $u \neq v$?

Cliques: Exercise (slido)



How many nodes has the largest clique of this graph?

CLIQUE is NP-Complete (1)

Theorem (CLIQUE is NP-Complete)

CLIQUE *is NP-complete.*

Clique is NP-Complete (2)

Proof.

Clique $\in$ NP: guess and check.

Clique is NP-hard: We show $\mathrm{3SAT} \leq_{\mathrm{p}}$ Clique.

- We are given a 3-CNF formula φ, and we may assume that each clause has exactly three literals.
- In polynomial time, we must construct a graph $G = \langle V, E \rangle$ and a number K such that: G has a clique of size at least K iff φ is satisfiable.

$\rightsquigarrow$ construction of V, E, K on the following slides.

...

CLIQUE is NP-Complete (3)

Proof (continued).

Let m be the number of clauses in φ.

Let ℓ_{ij} the j-th literal in clause i.

Define V, E, K as follows:

- $V = \{\langle i, j\rangle \mid 1 \leq i \leq m, 1 \leq j \leq 3\}$
  $\rightsquigarrow$ a vertex for every literal of every clause
- E contains edge between $\langle i, j\rangle$ and $\langle i', j'\rangle$ if and only if
  - $i \neq i'$ $\rightsquigarrow$ belong to different clauses, and
  - ℓ_{ij} and $\ell_{i'j'}$ are not complementary literals
- $K = m$

$\rightsquigarrow$ obviously polynomially computable

to show: reduction property . . .

CLIQUE is NP-Complete (4)

Proof (continued).

($\Rightarrow$): If φ is satisfiable, then $\langle V, E\rangle$ has clique of size at least K:

- Given a satisfying variable assignment choose a vertex corresponding to a satisfied literal in each clause.
- The chosen K vertices are all connected with each other and hence form a clique of size K.

...

Clique is NP-Complete (5)

Proof (continued).

($\Leftarrow$): If $\langle V, E\rangle$ has a clique of size at least K, then φ is satisfiable:

- Consider a given clique C of size at least K.
- The vertices in C must all correspond to different clauses (vertices in the same clause are not connected by edges).

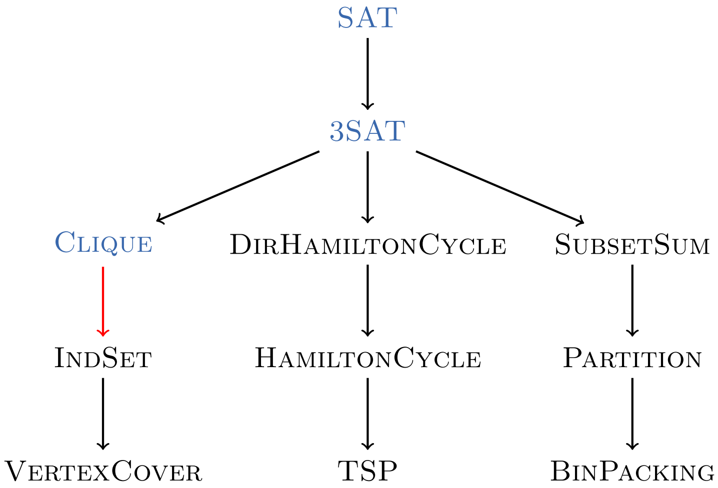
$\rightsquigarrow$ exactly one vertex per clause is included in C

- Two vertices in C never correspond to complementary literals X and $\neg X$ (due to the way we defined the edges).
- If a vertex corresp. to X was chosen, map X to T (true).
- If a vertex corresp. to $\neg X$ was chosen, map X to F (false).
- If neither was chosen, arbitrarily map X to T or F.

$\rightsquigarrow$ satisfying assignment

$\square$

Clique $\leq_p$ IndSet



IndSet

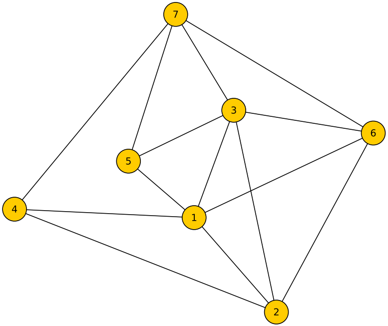
Definition (IndSet)

The problem IndSet is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have an independent set of size at least K, i. e., a set of vertices $I \subseteq V$ with $|I| \geq K$ and $\{u, v\} \notin E$ for all $u, v \in I$ with $u \neq v$?

Independent Set: Exercise (slido)



Does this graph have an independent set of size 3?

IndSet is NP-Complete (1)

Theorem (IndSet is NP-Complete)

IndSet *is NP-complete.*

Proof.

IndSet $\in$ NP: guess and check. . . .

IndSet is NP-Complete (2)

Proof (continued).

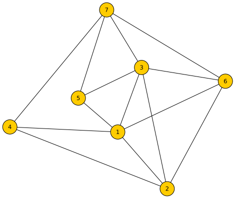
IndSet is NP-hard: We show Clique $\leq_p$ IndSet.

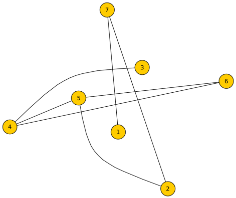
We describe a polynomial reduction f.

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for Clique.

Then $f(\langle G, K \rangle)$ is the IndSet instance $\langle \overline{G}, K \rangle$, where $\overline{G} := \langle V, \overline{E} \rangle$ and $\overline{E} := \{\{u, v\} \subseteq V \mid u \neq v, \{u, v\} \notin E\}$.
(This graph $\overline{G}$ is called the complement graph of G.)

Clearly f can be computed in polynomial time. ...





$\textsc{IndSet}$ is NP-Complete (3)

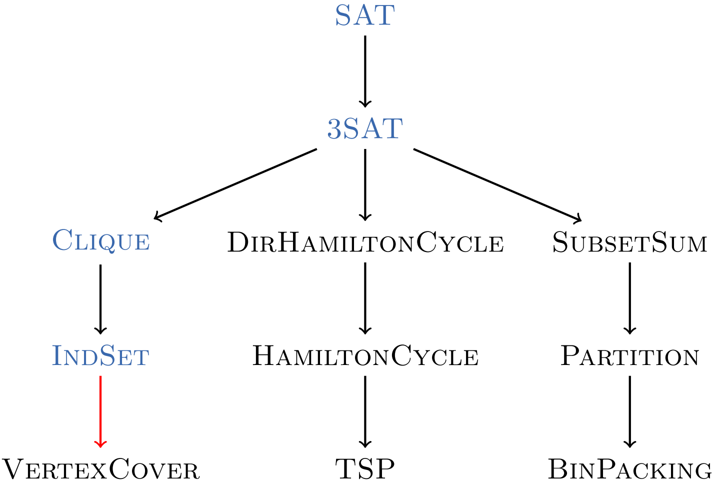
Proof (continued).

We have:

$$
\begin{aligned}
&\langle\langle V, E\rangle, K\rangle \in \textsc{Clique} \\
\text{iff}\quad & \text{there exists a set } V' \subseteq V \text{ with } |V'| \geq K \\
& \quad \text{and } \{u, v\} \in E \text{ for all } u, v \in V' \text{ with } u \neq v \\
\text{iff}\quad & \text{there exists a set } V' \subseteq V \text{ with } |V'| \geq K \\
& \quad \text{and } \{u, v\} \notin \overline{E} \text{ for all } u, v \in V' \text{ with } u \neq v \\
\text{iff}\quad & \langle\langle V, \overline{E}\rangle, K\rangle \in \textsc{IndSet} \\
\text{iff}\quad & f(\langle\langle V, E\rangle, K\rangle) \in \textsc{IndSet}
\end{aligned}
$$

and hence f is a reduction. □

IndSet $\leq_p$ VertexCover



VertexCover

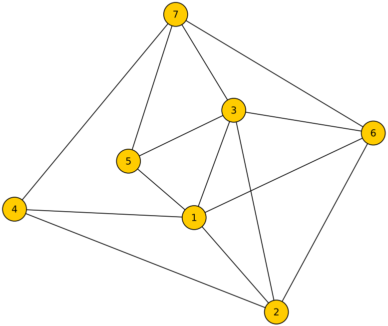
Definition (VertexCover)

The problem VertexCover is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a vertex cover of size at most K, i. e., a set of vertices $C \subseteq V$ with $|C| \leq K$ and $\{u, v\} \cap C \neq \emptyset$ for all $\{u, v\} \in E$?

Vertex Cover: Exercise (slido)



Does this graph have a vertex cover of size 4?

VertexCover is NP-Complete (1)

Theorem (VertexCover is NP-Complete)

VertexCover *is NP-complete.*

VertexCover is NP-Complete (2)

Proof.

VertexCover $\in$ NP: guess and check.

VertexCover is NP-hard:

We show IndSet $\leq_p$ VertexCover.

We describe a polynomial reduction f.

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for IndSet.

Then $f(\langle G, K \rangle) := \langle G, |V| - K \rangle$.

This can clearly be computed in polynomial time. . . .

VertexCover is NP-Complete (3)

Proof (continued).

For vertex set $V' \subseteq V$, we write $\overline{V'}$ for its complement $V \setminus V'$.

Observation: a set of vertices is a vertex cover iff its complement is an independent set.

We thus have:

$$
\begin{aligned}
& \langle\langle V, E\rangle, K\rangle \in \textsc{IndSet} \\
\text{iff}\quad & \langle V, E\rangle \text{ has an independent set } I \text{ with } |I| \geq K \\
\text{iff}\quad & \langle V, E\rangle \text{ has a vertex cover } C \text{ with } |\overline{C}| \geq K \\
\text{iff}\quad & \langle V, E\rangle \text{ has a vertex cover } C \text{ with } |C| \leq |V| - K \\
\text{iff}\quad & \langle\langle V, E\rangle, |V| - K\rangle \in \textsc{VertexCover} \\
\text{iff}\quad & f(\langle\langle V, E\rangle, K\rangle) \in \textsc{VertexCover}
\end{aligned}
$$

and hence f is a reduction. □

D4.2 Routing Problems

3SAT $\leq_p$ DirHamiltonCycle

$$
\begin{tikzcd}
& \text{SAT} \arrow[d] & \\
& \text{3SAT} \arrow[dl] \arrow[d] \arrow[dr] & \\
\textsc{Clique} \arrow[d] & \textsc{DirHamiltonCycle} \arrow[d] & \textsc{SubsetSum} \arrow[d] \\
\textsc{IndSet} \arrow[d] & \textsc{HamiltonCycle} \arrow[d] & \textsc{Partition} \arrow[d] \\
\textsc{VertexCover} & \text{TSP} & \textsc{BinPacking}
\end{tikzcd}
$$

DirHamiltonCycle is NP-Complete (1)

Definition (Reminder: DirHamiltonCycle)

The problem DirHamiltonCycle is defined as follows:

Given: directed graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

DirHamiltonCycle *is NP-complete.*

DirHamiltonCycle is NP-Complete (2)

Proof.

DirHamiltonCycle $\in$ NP: guess and check.

DirHamiltonCycle is NP-hard:

We show $3\mathrm{SAT} \leq_{\mathsf{p}}$ DirHamiltonCycle.

- We are given a 3-CNF formula φ where each clause contains exactly three literals and no clause contains duplicated literals.
- We must, in polynomial time, construct a directed graph $G = \langle V, E \rangle$ such that: G contains a Hamilton cycle iff φ is satisfiable.
- construction of $\langle V, E \rangle$ on the following slides

...

DirHamiltonCycle is NP-Complete (3)

Proof (continued).

- Let $X_1, \dots, X_n$ be the atomic propositions in φ.
- Let $c_1, \dots, c_m$ be the clauses of φ with $c_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$.
- Construct a graph with $6m + n$ vertices (described on the following slides).

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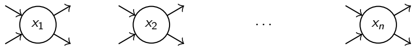
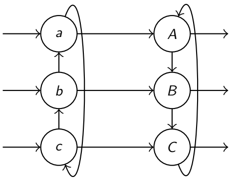
DirHamiltonCycle is NP-Complete (4)

Proof (continued).

- For every variable X_i, add vertex x_i
  with 2 incoming and 2 outgoing edges:



- For every clause c_j, add the subgraph C_j with 6 vertices:



- We describe later how to connect these parts.

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DirHamiltonCycle is NP-Complete (5)

Proof (continued).

Let π be a Hamilton cycle of the total graph.

- Whenever π enters subgraph C_j from one of its "entrances", it must leave via the corresponding "exit": $(a \longrightarrow A,\ b \longrightarrow B,\ c \longrightarrow C)$. Otherwise, π cannot be a Hamilton cycle.
- Hamilton cycles can behave in the following ways with regard to C_j:
  - π passes through C_j once (from any entrance)
  - π passes through C_j twice (from any two entrances)
  - π passes through C_j three times (once from every entrance)

...

DirHamiltonCycle is NP-Complete (6)

Proof (continued).

Connect the "open ends" in the graph as follows:

- Identify entrances/exits of the clause subgraph C_j with the three literals in clause c_j.
- One exit of x_i is positive, the other one is negative.
- For the positive exit, determine the clauses in which the positive literal X_i occurs:
  - Connect the positive exit of x_i with the X_i-entrance of the first such clause graph.
  - Connect the X_i-exit of this clause graph with the X_i-entrance of the second such clause graph, and so on.
  - Connect the X_i-exit of the last such clause graph with the positive entrance of x_{i+1} (or x_1 if $i = n$).
- analogously for the negative exit of x_i and the literal $\neg X_i$

...

DirHamiltonCycle is NP-Complete (7)

Proof (continued).

The construction is polynomial and is a reduction:

(⇒): construct a Hamilton cycle from a satisfying assignment

- Given a satisfying assignment $\mathcal{I}$, construct a Hamilton cycle that leaves x_i through the positive exit if $\mathcal{I}(X_i)$ is true and by the negative exit if $\mathcal{I}(X_i)$ is false.
- Afterwards, we visit all C_j-subgraphs for clauses that are satisfied by this literal.
- In total, we visit each C_j-subgraph 1–3 times.

...

DirHamiltonCycle is NP-Complete (8)

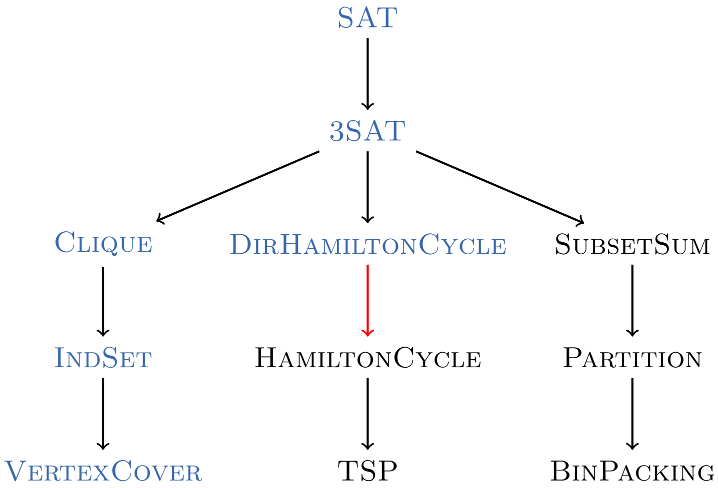
Proof (continued).

($\Leftarrow$): construct a satisfying assignment from a Hamilton cycle

- A Hamilton cycle visits every vertex x_i and leaves it by the positive or negative exit.
- Map X_i to true or false depending on which exit is used to leave x_i.
- Because the cycle must traverse each C_j-subgraph at least once (otherwise it is not a Hamilton cycle), this results in a satisfying assignment. (Details omitted.)

$\square$

DirHamiltonCycle $\leq_p$ HamiltonCycle



HamiltonCycle is NP-Complete (1)

Definition (Reminder: HamiltonCycle)

The problem **HamiltonCycle** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

HamiltonCycle *is NP-complete.*

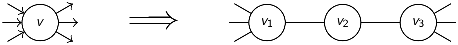
HamiltonCycle is NP-Complete (2)

Proof sketch.

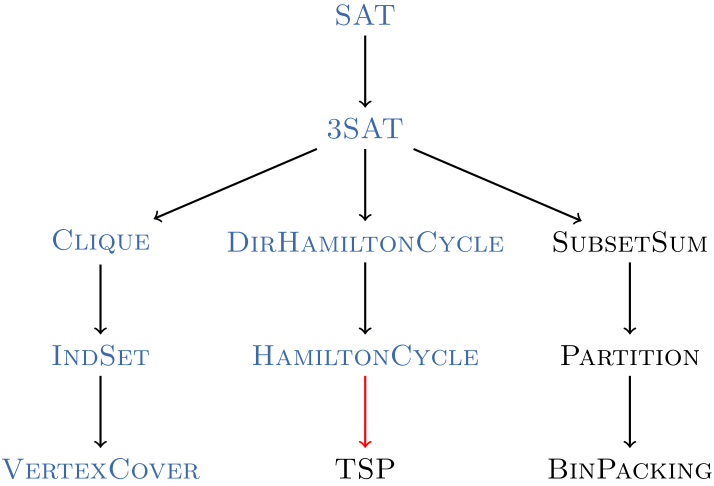
HamiltonCycle $\in$ NP: guess and check.

HamiltonCycle is NP-hard: We show DirHamiltonCycle $\leq_p$ HamiltonCycle.

Basic building block of the reduction:



HamiltonCycle $\leq_p$ TSP



TSP is NP-Complete (1)

Definition (Reminder: TSP)

TSP (traveling salesperson problem) is the following decision problem:

- Given: finite set $S \neq \emptyset$ of cities, symmetric cost function $cost : S \times S \to \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- Question: Is there a tour with total cost at most K, i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with $\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K$?

Theorem

TSP *is NP-complete.*

TSP is NP-Complete (2)

Proof.

TSP $\in$ NP: guess and check.

TSP is NP-hard: We showed HamiltonCycle $\leq_p$ TSP in Chapter D2. □

Summary

- In this chapter we showed NP-completeness of
  - three classical graph problems:
    CLIQUE, INDSET, VERTEXCOVER
  - three classical routing problems:
    DIRHAMILTONCYCLE, HAMILTONCYCLE, TSP