Theory of Computer Science

D3. Proving NP-Completeness

Gabriele Röger

University of Basel

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Overview

Reminder: P and NP

P: class of languages that are decidable in polynomial time by a deterministic Turing machine

NP: class of languages that are decidable in polynomial time by a non-deterministic Turing machine

Reminder: Polynomial Reductions

Definition (Polynomial Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be decision problems.
We say that A can be polynomially reduced to B,
written $A \leq_{\mathrm{p}} B$, if there is a function $f : \Sigma^* \to \Gamma^*$ such that:

- f can be computed in polynomial time by a DTM
- f reduces A to B
  - i. e., for all $w \in \Sigma^*$: $w \in A$ iff $f(w) \in B$

f is called a polynomial reduction from A to B

Transitivity of $\leq_{\mathrm{p}}$: If $A \leq_{\mathrm{p}} B$ and $B \leq_{\mathrm{p}} C$, then $A \leq_{\mathrm{p}} C$.

Reminder: NP-Hardness and NP-Completeness

Definition (NP-Hard, NP-Complete)

Let B be a decision problem.

B is called NP-hard if $A \leq_{\mathrm{p}} B$ for all problems $A \in \mathrm{NP}$.

B is called NP-complete if $B \in \mathrm{NP}$ and B is NP-hard.

Proving NP-Completeness by Reduction

- Suppose we know one NP-complete problem
  (we will use satisfiability of propositional logic formulas).
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  that further problems are NP-complete.

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Theorem (Proving NP-Completeness by Reduction)

Let A and B be problems such that:

- *A is NP-hard, and*
- *$A \leq_{\mathrm{p}} B$.*

Then B is also NP-hard.
If furthermore $B \in \mathrm{NP}$, then B is NP-complete.

Proving NP-Completeness by Reduction: Proof

Proof.

First part: We must show $X \leq_{\mathrm{p}} B$ for all $X \in \mathrm{NP}$.

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Second part: follows directly by definition of NP-completeness. □

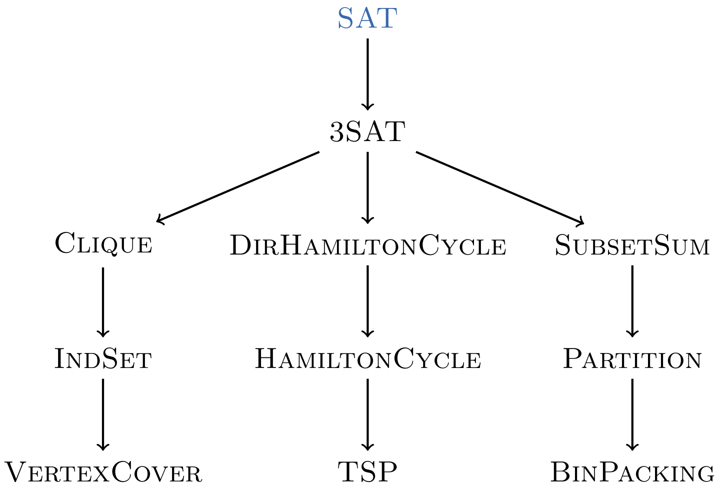
NP-Complete Problems

- There are thousands of known NP-complete problems.
- An extensive catalog of NP-complete problems from many areas of computer science is contained in:

  Michael R. Garey and David S. Johnson:
  Computers and Intractability —
  A Guide to the Theory of NP-Completeness
  W. H. Freeman, 1979.

- In the remaining chapters, we get to know some of these problems.

Overview of the Reductions

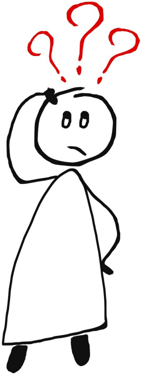


What Do We Have to Do?

- We want to show the NP-completeness of these 11 problems.
- We first show that SAT is NP-complete.
- Then it is sufficient to show
  - that polynomial reductions exist for all edges in the figure (and thus all problems are NP-hard)
  - and that the problems are all in NP.

(It would be sufficient to show membership in NP only for the leaves in the figure. But membership is so easy to show that this would not save any work.)

Questions



Questions?

Propositional Logic

- We need to establish NP-completeness of one problem "from scratch".
- We will use satisfiability of propositional logic formulas.
- So what is this?

Let's briefly cover the basics.

Propositional Logic: Syntax

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  $\rightarrow$ variables that can be true or false

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Example

$\neg(X \wedge (Y \vee \neg(Z \wedge Y)))$ is a propositional formula over $\{X, Y, Z\}$.

Propositional Logic: Semantics

- A truth assignment for a set of atomic propositions A is a function $\mathcal{I} : A \to \{T, F\}$.
- A formula can be true or false under a given truth assignment. Write $\mathcal{I} \models \varphi$ to express that φ is true under $\mathcal{I}$.
  - Atomic variable a is true under $\mathcal{I}$ iff $\mathcal{I}(a) = T$.
  - Negation $\neg\varphi$ is true under $\mathcal{I}$ iff φ is not: $\mathcal{I} \models \neg\varphi$ iff $\mathcal{I} \not\models \varphi$
  - Conjunction $(\varphi_1 \wedge \cdots \wedge \varphi_n)$ is true under $\mathcal{I}$ iff each φ_i is: $\mathcal{I} \models (\varphi_1 \wedge \cdots \wedge \varphi_n)$ iff $\mathcal{I} \models \varphi_i$ for all $i \in \{1, \ldots, n\}$
  - Disjunction $(\varphi_1 \vee \cdots \vee \varphi_n)$ is true under $\mathcal{I}$ iff some φ_i is: $\mathcal{I} \models (\varphi_1 \vee \cdots \vee \varphi_n)$ iff exists $i \in \{1, \ldots, n\}$ such that $\mathcal{I} \models \varphi_i$

Propositional Logic: Example

Consider truth assignment $\mathcal{I} = \{X \mapsto F, Y \mapsto T, Z \mapsto F\}$.

Is $\neg(X \wedge (Y \vee \neg(Z \wedge Y)))$ true under $\mathcal{I}$?

Propositional Logic: Exercise (slido)

Consider truth assignment

$$\mathcal{I} = \{X \mapsto F, Y \mapsto T, Z \mapsto F\}.$$

Is $(X \vee (\neg Z \wedge Y))$ true under $\mathcal{I}$?

More Propositional Logic

- $(\varphi \to \psi)$ is a short-hand notation for formula $(\neg\varphi \lor \psi)$.

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  - both, φ and ψ are true under $\mathcal{I}$, or
  - neither φ nor ψ is true under $\mathcal{I}$.

Short Notations for Conjunctions and Disjunctions

Short notation for addition:

$$\sum_{x \in \{x_1, \dots, x_n\}} x = x_1 + x_2 + \dots + x_n$$

Analogously (possible because of commutativity of $\wedge$ and $\vee$):

$$\left(\bigwedge_{\varphi \in X} \varphi\right) = (\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n)$$

$$\left(\bigvee_{\varphi \in X} \varphi\right) = (\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_n)$$

$$\text{for } X = \{\varphi_1, \dots, \varphi_n\}$$

SAT Problem

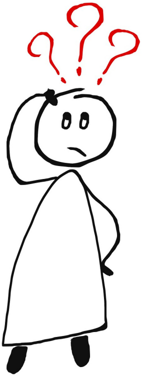
Definition (SAT)

The problem SAT (satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable,
i.e. is there a variable assignment $\mathcal{I}$ such that $\mathcal{I} \models \varphi$?

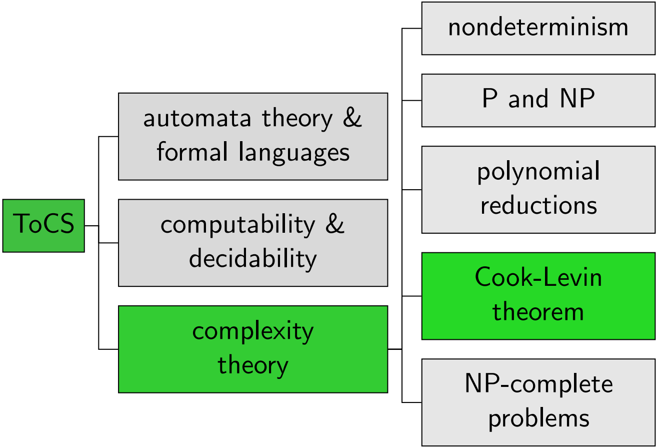
Questions



Questions?

Cook-Levin Theorem

Content of the Course



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Theorem (Cook, 1971; Levin, 1973)

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Theorem (Cook, 1971; Levin, 1973)

SAT *is NP-complete.*

Proof.

SAT $\in$ NP: guess and check.
SAT is NP-hard: somewhat more complicated (to be continued)

...

NP-hardness of SAT (1)

Proof (continued).

We must show: $A \leq_{\mathrm{p}} \mathrm{SAT}$ for all $A \in \mathrm{NP}$.

NP-hardness of SAT (1)

Proof (continued).

We must show: $A \leq_{\mathsf{p}} \mathrm{SAT}$ for all $A \in \mathsf{NP}$.

Let A be an arbitrary problem in NP.

We have to find a polynomial reduction of A to SAT,
i. e., a function f computable in polynomial time
such that for every input word w over the alphabet of A:

$w \in A$ iff $f(w)$ is a satisfiable propositional formula. ...

NP-hardness of SAT (2)

Proof (continued).

Because $A \in \mathrm{NP}$, there is an NTM M and a polynomial p such that M decides the problem A in time p.

Idea: construct a formula that encodes the possible configurations which M can reach in time $p(|w|)$ on input w
and that is satisfiable if and only if
an accepting configuration can be reached in this time. . . .

NP-hardness of SAT (3)

Proof (continued).

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ be an NTM for A, and let p be a polynomial bounding the computation time of M. Without loss of generality, $p(n) \geq n$ for all n.

Let $w = w_1 \ldots w_n \in \Sigma^*$ be the input for M.

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We number the tape positions with natural numbers such that the TM head initially is on position 1.

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Observation: within $p(n)$ computation steps the TM head can only reach positions in the set $Pos = \{1, \dots, p(n) + 1\}$.

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Observation: within $p(n)$ computation steps the TM head can only reach positions in the set $Pos = \{1, \dots, p(n) + 1\}$.

Instead of infinitely many tape positions, we now only need to consider these (polynomially many!) positions. ...

NP-hardness of SAT (4)

Proof (continued).

We can encode configurations of M by specifying:

- what the current state of M is
- on which position in Pos the TM head is located
- which symbols from Γ the tape contains at positions Pos

$\rightsquigarrow$ can be encoded by propositional variables

NP-hardness of SAT (4)

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We only need to consider the computation steps $Steps = \{0, 1, \dots, p(n)\}$ because M should accept within $p(n)$ steps. ...

NP-hardness of SAT (5)

Proof (continued).

Use the following propositional variables in formula $f(w)$:

- $\mathit{state}_{t,q}$ ($t \in \mathit{Steps}$, $q \in Q$)
  $\rightsquigarrow$ encodes the state of the NTM in the t-th configuration
- $\mathit{head}_{t,i}$ ($t \in \mathit{Steps}$, $i \in \mathit{Pos}$)
  $\rightsquigarrow$ encodes the head position in the t-th configuration
- $\mathit{tape}_{t,i,a}$ ($t \in \mathit{Steps}$, $i \in \mathit{Pos}$, $a \in \Gamma$)
  $\rightsquigarrow$ encodes the tape content in the t-th configuration

Construct $f(w)$ such that every satisfying interpretation

- describes a sequence of NTM configurations
- that begins with the start configuration,
- reaches an accepting configuration
- and follows the NTM rules in δ

...

NP-hardness of SAT (6)

Proof (continued).

Auxiliary formula:

$$\mathit{oneof}\, X := \left(\bigvee_{x \in X} x\right) \wedge \neg \left(\bigvee_{x \in X} \bigvee_{y \in X \setminus \{x\}} (x \wedge y)\right)$$

Auxiliary notation:

The symbol $\bot$ stands for an arbitrary unsatisfiable formula (e.g., $(A \wedge \neg A)$, where A is an arbitrary proposition). ...

NP-hardness of SAT (7)

Proof (continued).

1. describe the configurations of the TM:

$$
\begin{aligned}
\mathit{Valid} := \bigwedge_{t \in \mathit{Steps}} \Big(& \mathit{oneof}\, \{\mathit{state}_{t,q} \mid q \in Q\} \wedge \\
& \mathit{oneof}\, \{\mathit{head}_{t,i} \mid i \in \mathit{Pos}\} \wedge \\
& \bigwedge_{i \in \mathit{Pos}} \mathit{oneof}\, \{\mathit{tape}_{t,i,a} \mid a \in \Gamma\} \Big)
\end{aligned}
$$

...

NP-hardness of SAT (8)

Proof (continued).

2. begin in the start configuration

$$Init := state_{0,q_0} \wedge \mathrm{head}_{0,1} \wedge \bigwedge_{i=1}^{n} tape_{0,i,w_i} \wedge \bigwedge_{i \in Pos \setminus \{1,\dots,n\}} tape_{0,i,\square}$$

...

NP-hardness of SAT (9)

Proof (continued).

3. reach an accepting configuration

$$Accept := \bigvee_{t \in Steps} state_{t,q_{\text{accept}}}$$

...

NP-hardness of SAT (10)

Proof (continued).

4. follow the rules in δ:

$$Trans := \bigwedge_{t \in Steps} \left(state_{t,q_{\text{accept}}} \lor state_{t,q_{\text{reject}}} \lor \bigvee_{R \in \delta} Rule_{t,R} \right)$$

where... ...

NP-hardness of SAT (11)

Proof (continued).

4. follow the rules in δ (continued):

$$
\begin{aligned}
&\mathit{Rule}_{t,\langle\langle q,a\rangle,\langle q',a',D\rangle\rangle} := \\
&\qquad \mathit{state}_{t,q} \wedge \mathit{state}_{t+1,q'} \wedge \\
&\qquad \bigwedge_{i \in \mathit{Pos}} \left(\mathit{head}_{t,i} \to \left(\mathit{tape}_{t,i,a} \wedge \mathit{head}_{t+1,i+D} \wedge \mathit{tape}_{t+1,i,a'}\right)\right) \\
&\qquad \wedge \bigwedge_{i \in \mathit{Pos}} \bigwedge_{a'' \in \Gamma} \left(\left(\neg \mathit{head}_{t,i} \wedge \mathit{tape}_{t,i,a''}\right) \to \mathit{tape}_{t+1,i,a''}\right)
\end{aligned}
$$

- For $i + D$, interpret $i + \mathrm{R} \rightsquigarrow i + 1$, $i + \mathrm{L} \rightsquigarrow \max\{1, i - 1\}$.
- special case: *tape* and *head* variables with a tape index $i + D$ outside of *Pos* are replaced by $\perp$; likewise all variables with a time index outside of *Steps*.

...

NP-hardness of SAT (12)

Proof (continued).

Putting the pieces together:

Set $f(w) := \mathit{Valid} \land \mathit{Init} \land \mathit{Accept} \land \mathit{Trans}$.

NP-hardness of SAT (12)

Proof (continued).

Putting the pieces together:

Set $f(w) := \mathit{Valid} \wedge \mathit{Init} \wedge \mathit{Accept} \wedge \mathit{Trans}$.

- $f(w)$ can be constructed in time polynomial in $|w|$.
- $w \in A$ iff M accepts w in $p(|w|)$ steps
  iff $f(w)$ is satisfiable
  iff $f(w) \in \mathrm{SAT}$

$\rightsquigarrow$ $A \leq_{\mathrm{p}} \mathrm{SAT}$

NP-hardness of SAT (12)

Proof (continued).

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Since $A \in \mathrm{NP}$ was arbitrary, this is true for every $A \in \mathrm{NP}$.

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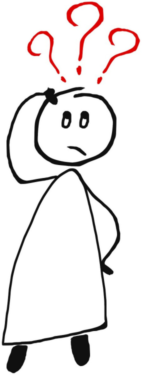
$\leadsto A \leq_{\mathsf{p}} \mathrm{SAT}$

Since $A \in \mathsf{NP}$ was arbitrary, this is true for every $A \in \mathsf{NP}$.
Hence SAT is NP-hard and thus also NP-complete. $\square$

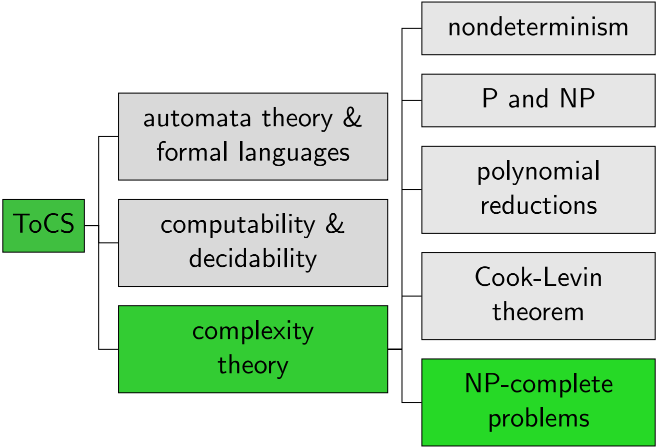
Questions



Questions?

3SAT

Content of the Course



More Propositional Logic: Conjunctive Normal Form

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More Propositional Logic: Conjunctive Normal Form

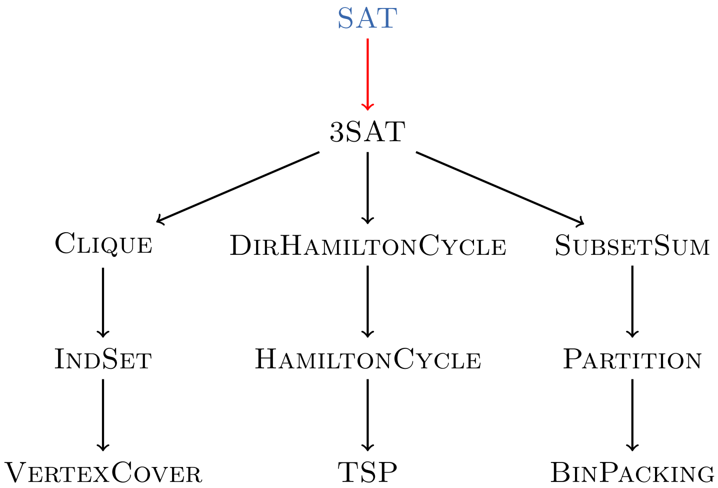
- A literal is an atomic proposition X or its negation $\neg X$.
- A clause is a disjunction of literals,
  e.g. $(X \vee \neg Y \vee Z)$
- A formula in conjunctive normal form
  is a conjunction of clauses,
  e.g. $((X \vee \neg Y \vee Z) \wedge (\neg X \vee \neg Z) \wedge (X \vee Y))$

Exercise (slido)

Which of the following formulas are in conjunctive normal form?

- $((X \wedge \neg Y \wedge Z) \vee (\neg X \wedge \neg Z))$
- $(X \vee \neg Y \vee Z)$
- $((\neg X \vee \neg Z) \wedge \neg(X \vee Y))$
- $((\neg Y \vee X) \wedge (Y \vee \neg Z))$

SAT $\leq_p$ 3SAT



SAT and 3SAT

Definition (Reminder: SAT)

The problem SAT (satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable?

Definition (3SAT)

The problem 3SAT is defined as follows:

Given: a propositional logic formula φ in conjunctive normal form with at most three literals per clause

Question: Is φ satisfiable?

3SAT is NP-Complete (1)

Theorem (3SAT is NP-Complete)

3SAT *is NP-complete.*

3SAT is NP-Complete (2)

Proof.

3SAT $\in$ NP: guess and check.

3SAT is NP-hard: We show SAT $\leq_p$ 3SAT.

- Let φ be the given input for SAT. Let *Sub*(φ) denote the set of subformulas of φ, including φ itself.

3SAT is NP-Complete (2)

Proof.

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- For each new proposition X_ψ, define the following auxiliary formula χ_ψ:
  - If $\psi = A$ for an atom A: $\chi_\psi = (X_\psi \leftrightarrow A)$
  - If $\psi = \neg\psi'$: $\chi_\psi = (X_\psi \leftrightarrow \neg X_{\psi'})$
  - If $\psi = (\psi' \wedge \psi'')$: $\chi_\psi = (X_\psi \leftrightarrow (X_{\psi'} \wedge X_{\psi''}))$
  - If $\psi = (\psi' \vee \psi'')$: $\chi_\psi = (X_\psi \leftrightarrow (X_{\psi'} \vee X_{\psi''}))$

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- Consider the conjunction of all these auxiliary formulas, $\chi_{\mathsf{all}} := \bigwedge_{\psi \in \mathit{Sub}(\varphi)} \chi_\psi$.

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- Hence, this describes a polynomial-time reduction.

$\square$

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- a clause may not contain the same literal twice

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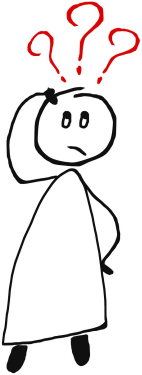
- fill up clauses with fewer than three literals with $\neg X$ and if necessary additionally with $\neg Y$

Questions



Questions?

Summary

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- Thousands of important problems are NP-complete.
- The satisfiability problem of propositional logic (SAT) is NP-complete.
- Proof idea for NP-hardness:
  - Every problem in NP can be solved by an NTM in polynomial time $p(|w|)$ for input w.
  - Given a word w, construct a propositional logic formula φ that encodes the computation steps of the NTM on input w.
  - Construct φ so that it is satisfiable if and only if there is an accepting computation of length $p(|w|)$.
- Usually (as seen for $3\mathrm{SAT}$), the easiest way to show that another problem is NP-complete is to
  - show that it is in NP with a guess-and-check algorithm, and
  - polynomially reduce a known NP-complete to it.