Theory of Computer Science

D1. Nondeterministic Algorithms, P and NP

Gabriele Röger

University of Basel

May 5, 2025

Overview: Course

contents of this course:

A. background ✓
  - ▷ mathematical foundations and proof techniques

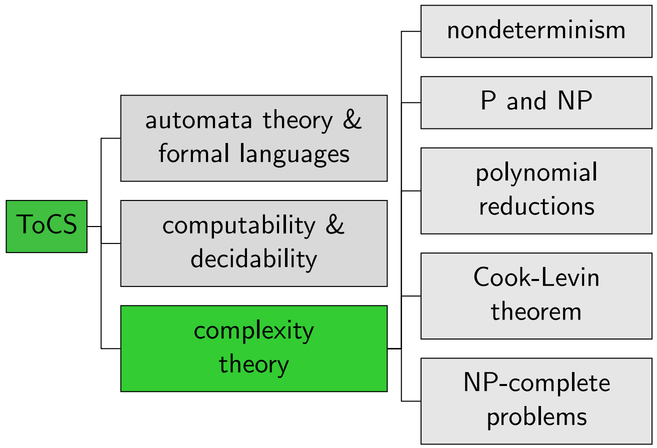
B. automata theory and formal languages ✓
  - ▷ What is a computation?

C. Turing computability ✓
  - ▷ What can be computed at all?

D. complexity theory
  - ▷ What can be computed efficiently?

E. more computability theory
  - ▷ Other models of computability

Content of the Course



D1.1 Motivation

A Scenario (1)

Example Scenario

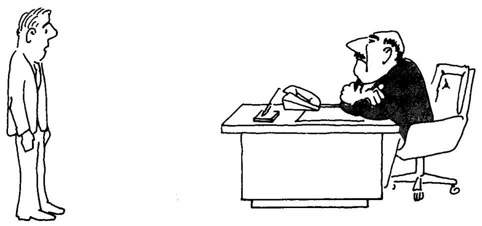
- You are a programmer at a logistics company.
- Your boss gives you the task of developing a program to optimize the route of a delivery truck:
  - The truck begins its route at the company depot.
  - It has to visit 50 stops.
  - You know the distances between all relevant locations (stops and depot).
  - Your program should compute a tour visiting all stops and returning to the depot on a shortest route.

A Scenario (2)

Example Scenario (ctd.)

- You work on the problem for weeks, but you do not manage to complete the task.
- All of your attempted programs
  - compute routes that are possibly suboptimal, or
  - do not terminate in reasonable time (say: within a month).
- What do you say to your boss?

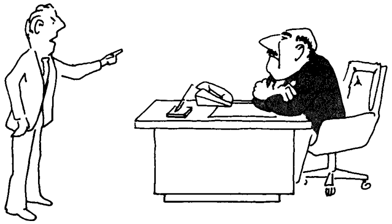
What You Don't Want to Say



"I can't find an efficient algorithm, I guess I'm just too dumb."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

What You Would Like to Say



"I can't find an efficient algorithm,
because no such algorithm is possible!"

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

What Complexity Theory Allows You to Say

"I can't find an efficient algorithm,
but neither can all these famous people."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 3

Why Complexity Theory?

Complexity Theory

Complexity theory tells us which problems can be solved quickly ("simple problems") and which ones cannot ("hard problems").

- This is useful in practice because simple and hard problems require different techniques to solve.
- If we can show that a problem is hard we do not need to waste our time with the (futile) search for a "simple" algorithm.

Test Your Intuition! (1)

- The following slide lists some graph problems.
- The input is always a directed graph $G = \langle V, E \rangle$.
- How difficult are the problems in your opinion?
- Sort the problems
  from easiest (= requires least amount of time to solve)
  to hardest (= requires most time to solve)
- no justification necessary, just follow your intuition!
- anonymous and not graded

Test Your Intuition! (2)

1. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with minimal length.
2. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with maximal length.
3. Determine whether G is strongly connected (every node is reachable from every other node).
4. Find a cycle (non-empty path from u to u for any $u \in V$; multiple visits of nodes are allowed).
5. Find a cycle that visits all nodes.
6. Find a cycle that visits a given node u.
7. Find a path that visits all nodes without repeating a node.
8. Find a path that uses all edges without repeating an edge.

D1.2 How to Measure Running Time?

How to Measure Running Time?

- Time complexity is a way to measure how much time it takes to solve a problem.
- How can we define such a measure appropriately?

Example Statements about Running Time

Example statements about running time:

- "Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."
- "With a 1 MiB input file, `sort` takes at most 1 second on a modern computer."
- "Quicksort is faster than sorting by insertion."
- "Sorting by insertion is slow."

$\rightsquigarrow$ Very different statements with different pros and cons.

Precise Statements vs. General Statements

Example Statement about Running Time
"Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."

advantage: very precise

disadvantage: not general

- input-specific:
  What if we want to sort other files?
- machine-specific:
  What happens on a different computer?
- even situation-specific:
  Will we get the same result tomorrow that we got today?

General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

1. General Inputs
Instead of concrete inputs, we talk about general types of input:

- Example: running time to sort an input of size n in the worst case
- Example: running time to sort an input of size n in the average case

here: running time for input size n in the worst case

General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

2. Ignoring Details
Instead of exact formulas for the running time we specify the order of magnitude:

- Example: instead of saying that we need time $\lceil 1.2n \log n \rceil - 4n + 100$, we say that we need time $O(n \log n)$.
- Example: instead of saying that we need time $O(n \log n)$, $O(n^2)$ or $O(n^4)$, we say that we need polynomial time.

here: What can be computed in polynomial time?

General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

3. Abstract Cost Measures
Instead of the running time on a concrete computer we consider a more abstract cost measure:

- Example: count the number of executed machine code statements
- Example: count the number of executed Java byte code statements
- Example: count the number of element comparisons of a sorting algorithms

here: count the computation steps of a Turing machine (polynomially equivalent to other measures)

D1.3 Decision Problems

Decision Problems

- As before, we simplify our investigation by restricting our attention to decision problems.
- More complex computational problems can be solved with multiple queries for an appropriately defined decision problem ("playing 20 questions").
- Formally, decision problems are languages (as before), but we use an informal "given" / "question" notation where possible.

Example: Decision vs. General Problem (1)

Definition (Hamilton Cycle)

Let $G = \langle V, E \rangle$ be a (directed or undirected) graph.

A Hamilton cycle of G is a sequence of vertices in V, $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- π is a path: there is an edge from v_i to v_{i+1} for all $0 \leq i < n$
- π is a cycle: $v_0 = v_n$
- π is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- π is Hamiltonian: all nodes of V are included in π

Example: Decision vs. General Problem (2)

Example (Hamilton Cycles in Directed Graphs)

$\mathcal{P}$: general problem DirHamiltonCycleGen

- Input: directed graph $G = \langle V, E \rangle$
- Output: a Hamilton cycle of G or a message that none exists

$\mathcal{D}$: decision problem DirHamiltonCycle

- Given: directed graph $G = \langle V, E \rangle$
- Question: Does G contain a Hamilton cycle?

These problems are polynomially equivalent:
from a polynomial algorithm for one of the problems
one can construct a polynomial algorithm for the other problem.
(Without proof.)

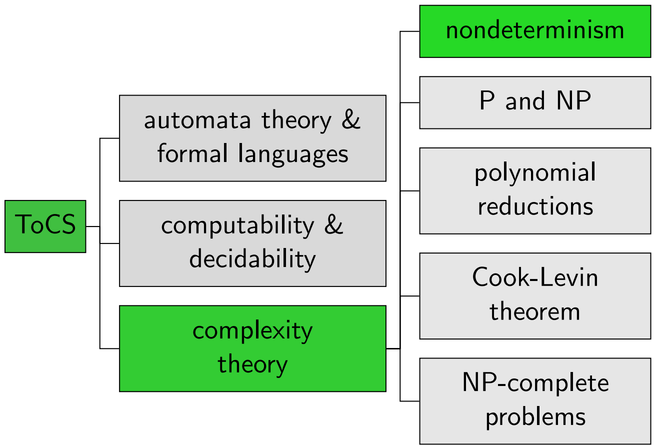
Algorithms for Decision Problems

Algorithms for decision problems:

- Where possible, we specify algorithms for decision problems in pseudo-code.
- Since they are only yes/no questions, we do not have to return a general result.
- Instead we use the statements
  - **ACCEPT** to accept the given input ("yes" answer) and
  - **REJECT** to reject it ("no" answer).
- Where we must be more formal, we use Turing machines and the notion of accepting from chapter B11.

D1.4 Nondeterminism

Content of the Course



Nondeterminism

- To develop complexity theory, we need the algorithmic concept of nondeterminism.
- already known for Turing machines (⇝ chapter B11):
  - An NTM can have more than one possible successor configuration for a given configuration.
  - Input x is accepted if there is at least one possible computation (configuration sequence) that leads to the accept state.
- Here we analogously introduce nondeterminism for pseudo-code.

Nondeterministic Algorithms

nondeterministic algorithms:

- All constructs of deterministic algorithms are also allowed in nondeterministic algorithms: **IF**, **WHILE**, etc.
- Additionally, there is a nondeterministic assignment:
  GUESS $x_i \in \{0, 1\}$
  where x_i is a program variable.

Nondeterministic Algorithms: Acceptance

- Meaning of **GUESS** $x_i \in \{0, 1\}$:
  x_i is assigned either the value 0 or the value 1.
- This implies that the behavior of the program on a given input is no longer uniquely defined: there are multiple possible execution paths.
- The program accepts a given input if at least one execution path leads to an **ACCEPT** statement.
- Otherwise, the input is rejected.

Note: asymmetry between accepting and rejecting!
(cf. Turing-recognizability)

More Complex GUESS Statements

- We will also guess more than one bit at a time:
  GUESS $x \in \{1, 2, \dots, n\}$
  or more generally
  GUESS $x \in S$
  for a finite set S.
- These are abbreviations and can be split into $\lceil \log_2 n \rceil$ (or $\lceil \log_2 |S| \rceil$) "atomic" **GUESS** statements.

Example: Nondeterministic Algorithms (1)

Example (DirHamiltonCycle)

input: directed graph $G = \langle V, E \rangle$

```
start := an arbitrary node from V
current := start
remaining := V \ {start}
WHILE remaining ≠ ∅:
    GUESS next ∈ remaining
    IF ⟨current, next⟩ ∉ E:
        REJECT
    remaining := remaining \ {next}
    current := next
IF ⟨current, start⟩ ∈ E:
    ACCEPT
ELSE:
    REJECT
```

Example: Nondeterministic Algorithms (2)

- With appropriate data structures, this algorithm solves the problem in $O(n \log n)$ program steps, where $n = |V| + |E|$ is the size of the input.
- How many steps would a deterministic algorithm need?

Guess and Check

- The DirHamiltonCycle example illustrates a general design principle for nondeterministic algorithms:

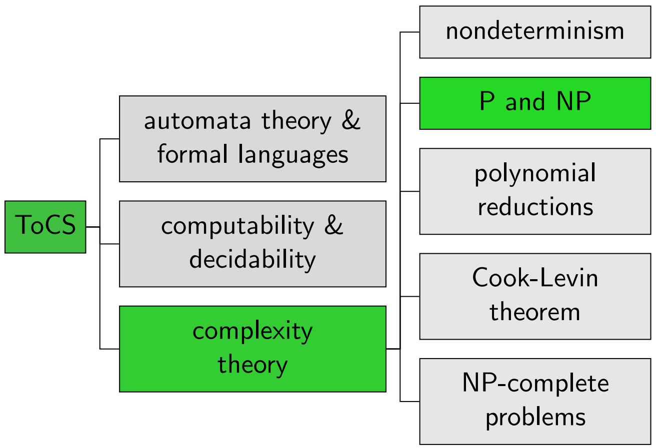
  guess and check
- In general, nondeterministic algorithms can solve a problem by first guessing a "solution" and then verifying that it is indeed a solution. (In the example, these two steps are interleaved.)
- If solutions to a problem can be efficiently verified, then the problem can also be efficiently solved if nondeterminism may be used.

The Power of Nondeterminism

- Nondeterministic algorithms are very powerful because they can "guess" the "correct" computation step.
- Or, interpreted differently: they go through many possible computations "in parallel", and it suffices if one of them is successful.
- Can they solve problems efficiently (in polynomial time) which deterministic algorithms cannot solve efficiently?
- This is the big question!

D1.5 P and NP

Content of the Course



Impact of Nondeterminism?

- We earlier established that deterministic and nondeterministic Turing machines recognize the same class of languages.
  $\rightarrow$ For this aspect, nondeterminism did not make a difference.
- Now we consider what decision problems can be solved in polynomial time.
- Does it make a difference whether we allow nondeterminism?

This is the famous P vs. NP question!

Running Time of a Deterministic Turing Machine

Definition (Running Time of a DTM)
Let M be a DTM that halts on all inputs. The running time or time complexity of M if the function $f : \mathbb{N} \rightarrow \mathbb{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n.

We say that
- M runs in time f and that
- M is an f time Turing machine.

Big-O

Definition (Big-O)
Let f and g be functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$.
We say that $f \in O(g)$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$

$$f(n) \leq cg(n).$$

Complexity Class P

Definition (Time Complexity Class TIME)

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function.

Define the time complexity class TIME(t(n)) to be the collection of all languages that are decidable by an $O(t)$ time Turing machine.

Definition (P)

P is the class of languages that are decidable in polynomial time by a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

Running Time of a Nondeterministic Turing Machine

Definition (Running Time of a NTM)

Let M be a NTM that is a decider, i. e. all its computation branches halt on all inputs.

The running time or time complexity of M if the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps that M uses on any branch of its computation on any input of length n.

Complexity Class NP

Definition (Time Complexity Class NTIME)

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function.

Define the time complexity class NTIME(t(n)) to be the collection of all languages that are decidable by an $O(t)$ time nondeterministic Turing machine.

Definition (NP)

NP is the class of languages that are decidable in polynomial time by a nondeterministic single-tape Turing machine. In other words,

$$\mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k).$$

P and NP: Remarks

- Sets of languages like P and NP that are defined in terms of computation time of TMs (or other computation models) are called complexity classes.
- We know that $\mathrm{P} \subseteq \mathrm{NP}$. (Why?)
- Whether the converse is also true is an open question: this is the famous P-NP problem.

Example: DirHamiltonCycle ∈ NP

Example (DirHamiltonCycle ∈ NP)
The nondeterministic algorithm of the previous section solves the problem and can be implemented on an NTM in polynomial time.

- Is DirHamiltonCycle ∈ P also true?
- The answer is unknown.
- So far, only exponential deterministic algorithms for the problem are known.

Simulation of NTMs with DTMs

- Unlike DTMs, NTMs are not a realistic computation model: they cannot be directly implemented on computers.
- But NTMs can be simulated by systematically trying all computation paths, e. g., with a breadth-first search.

More specifically:

- Let M be an NTM that decides language L in time f, where $f(n) \geq n$ for all $n \in \mathbb{N}_0$.
- Then we can specify a DTM M' that decides L in time f', where $f'(n) = 2^{O(f(n))}$.
- without proof
  (cf. "Introduction to the Theory of Computation" by Michael Sipser (3rd edition), Theorem 7.11)

Summary (1)

- Complexity theory deals with the question which problems can be solved efficiently and which ones cannot.
- here: focus on what can be computed in polynomial time
- To formalize this, we use Turing machines, but other formalisms are polynomially equivalent.
- We consider decision problems, but the results often directly transfer to general computational problems.

Summary (2)

important concept: nondeterminism

- Nondeterministic algorithms can "guess", i. e., perform multiple computations "at the same time".
- An input receives a "yes" answer if at least one computation path accepts it.
- in NTMs: with nondeterministic transitions ($\delta(q, a)$ contains multiple elements)
- in pseudo-code: with **GUESS** statements

Summary (3)

- P: languages decidable by DTMs in polynomial time
- NP: languages decidable by NTMs in polynomial time
- P $\subseteq$ NP but it is an open question whether P = NP.