Theory of Computer Science

C5. Post Correspondence Problem

Gabriele Röger

University of Basel

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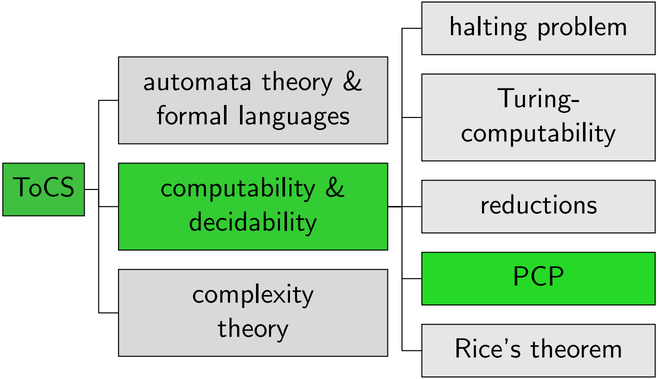
C5.1 Post Correspondence Problem

Content of the Course



More Options for Reduction Proofs?

- We can prove the undecidability of a problem with a reduction from an undecidable problem.
- The halting problem and the halting problem on the empty tape are possible options for this.
- both halting problem variants are quite similar ☹

→ We want a wider selection for reduction proofs
→ Is there some problem that is different in flavor?

Post correspondence problem
(named after mathematician Emil Leon Post)

Post Correspondence Problem: Example

Example (Post Correspondence Problem)

Given: different kinds of "dominos"

$$1{:}\ \begin{array}{|l|}\hline 1 \\ \hline 101 \\ \hline \end{array} \qquad 2{:}\ \begin{array}{|l|}\hline 10 \\ \hline 00 \\ \hline \end{array} \qquad 3{:}\ \begin{array}{|l|}\hline 011 \\ \hline 11 \\ \hline \end{array}$$

(an infinite number of each kind)

Question: Is there a sequence of dominos such that the upper and lower row match (= are equal)

$$\begin{array}{cccc} \begin{array}{|l|}\hline 1 \\ \hline 101 \\ \hline \end{array} & \begin{array}{|l|}\hline 011 \\ \hline 11 \\ \hline \end{array} & \begin{array}{|l|}\hline 10 \\ \hline 00 \\ \hline \end{array} & \begin{array}{|l|}\hline 011 \\ \hline 11 \\ \hline \end{array} \\ 1 & 3 & 2 & 3 \end{array}$$

Post Correspondence Problem: Definition

Definition (Post Correspondence Problem PCP)

Given: Finite sequence of pairs of words $(t_1, b_1), (t_2, b_2), \dots, (t_k, b_k)$, where $t_i, b_i \in \Sigma^+$ (for an arbitrary alphabet Σ)

Question: Is there a sequence $i_1, i_2, \dots, i_n \in \{1, \dots, k\}$, $n \geq 1$, with $t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}$?

A solution of the correspondence problem is such a sequence $i_1, \dots, i_n$, which we call a match.

Exercise (slido)

Consider PCP instance $(11, 1), (0, 00), (10, 01), (01, 11)$.

Is $2, 4, 3, 3, 1$ a match?

Given-Question Form vs. Definition as Set

So far: problems defined as sets
Now: definition in Given-Question form

Definition (new problem P)

Given: Instance $\mathcal{I}$
Question: Does $\mathcal{I}$ have a specific property?

corresponds to definitions

Definition (new problem P)

The problem P is the language
$\mathrm{P} = \{w \mid w \text{ encodes an instance } \mathcal{I} \text{ with the required property}\}$.

Definition (new problem P)

The problem P is the language
$\mathrm{P} = \{\langle\!\langle \mathcal{I} \rangle\!\rangle \mid \mathcal{I} \text{ is an instance with the required property}\}$.

PCP Definition as Set

We can alternatively define PCP as follows:

Definition (Post Correspondence Problem PCP)

The Post Correspondence Problem PCP is the set

$$\begin{aligned}\mathrm{PCP} = \{w \mid\ & w \text{ encodes a sequence of pairs of words} \\ & (t_1, b_1), (t_2, b_2), \dots, (t_k, b_k), \text{ for which} \\ & \text{there is a sequence } i_1, i_2, \dots, i_n \in \{1, \dots, k\} \\ & \text{such that } t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}\}.\end{aligned}$$

C5.2 (Un-)Decidability of PCP

Post Correspondence Problem

PCP cannot be so hard, huh?

– Is it?

1101	0110	1
1	11	110

Formally: $K = ((1101, 1), (0110, 11), (1, 110))$
$\rightarrow$ Shortest match has length 252!

10	0	100
0	001	1

Formally: $K = ((10, 0), (0, 001), (100, 1))$
$\rightarrow$ Unsolvable

PCP: Turing-recognizability

Theorem (Turing-recognizability of PCP)

PCP *is Turing-recognizable.*

Proof.

Recognition procedure for input w:

- If w encodes a sequence $(t_1, b_1), \dots, (t_k, b_k)$ of pairs of words:
  Test systematically longer and longer sequences $i_1, i_2, \dots, i_n$ whether they represent a match.
  If yes, terminate and return "yes".
- If w does not encode such a sequence: enter an infinite loop.

If $w \in \mathrm{PCP}$ then the procedure terminates with "yes", otherwise it does not terminate. □

PCP: Undecidability

Theorem (Undecidability of PCP)
PCP *is undecidable.*

Proof via an intermediate other problem
modified PCP (MPCP)

1. Reduce MPCP to PCP (MPCP $\leq$ PCP)
2. Reduce halting problem to MPCP ($H \leq$ MPCP)

$\rightarrow$ Let's get started...

MPCP: Definition

Definition (Modified Post Correspondence Problem MPCP)

Given: Sequence of word pairs as for PCP

Question: Is there a match $i_1, i_2, \ldots, i_n \in \{1, \ldots, k\}$ with $i_1 = 1$?

Reducibility of MPCP to PCP(1)

Lemma

MPCP $\leq$ PCP.

Proof.

Let $\#, \$ \notin \Sigma$. For word $w = a_1 a_2 \dots a_m \in \Sigma^+$ define

$$\bar{w} = \#a_1\#a_2\# \dots \#a_m\#$$
$$\grave{w} = \#a_1\#a_2\# \dots \#a_m$$
$$\acute{w} = a_1\#a_2\# \dots \#a_m\#$$

For input $C = ((t_1, b_1), \dots, (t_k, b_k))$ define
$f(C) = ((\bar{t_1}, \grave{b_1}), (\acute{t_1}, \grave{b_1}), (\acute{t_2}, \grave{b_2}), \dots, (\acute{t_k}, \grave{b_k}), (\$, \#\$))$

...

Reducibility of MPCP to PCP(2)

Proof (continued).

$f(C) = ((\bar{t_1}, \grave{b_1}), (\acute{t_1}, \grave{b_1}), (\acute{t_2}, \grave{b_2}), \dots, (\acute{t_k}, \grave{b_k}), (\$, \#\$))$

Function f is computable, and can suitably get extended to a total function. It holds that
C has a solution with $i_1 = 1$ iff $f(C)$ has a solution:

Let $1, i_2, i_3, \dots, i_n$ be a solution for C. Then
$1, i_2 + 1, \dots, i_n + 1, k + 2$ is a solution for $f(C)$.

If $i_1, \dots, i_n$ is a match for $f(C)$, then (due to the construction of the word pairs) there is a $m \leq n$ such that $i_1 = 1, i_m = k + 2$ and $i_j \in \{2, \dots, k + 1\}$ for $j \in \{2, \dots, m - 1\}$. Then
$1, i_2 - 1, \dots, i_{m-1} - 1$ is a solution for C.

$\Rightarrow$ f is a reduction from MPCP to PCP. $\square$

PCP: Undecidability – Where are we?

Theorem (Undecidability of PCP)
PCP *is undecidable.*

Proof via an intermediate other problem
modified PCP (MPCP)

1. Reduce MPCP to PCP (MPCP $\leq$ PCP) ✓
2. Reduce halting problem to MPCP ($H \leq$ MPCP)

Reducibility of H to MPCP (1)

Lemma

$H \leq \mathrm{MPCP}$.

Proof.

Goal: Construct for Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{accept}}, q_{\mathrm{reject}} \rangle$ and word $w \in \Sigma^*$ an MPCP instance $C = ((t_1, b_1), \ldots, (t_k, b_k))$ such that

$$M \text{ started on } w \text{ terminates iff } C \in \mathrm{MPCP}.$$

...

Reducibility of H to MPCP(2)

Proof (continued).

Idea:

- Sequence of words describes sequence of configurations of the TM
- "t-row" follows "b-row"

x : $\#\ c_0\ \#\ c_1\ \#\ c_2\ \#$

y : $\#\ c_0\ \#\ c_1\ \#\ c_2\ \#\ c_3\ \#$

- Configurations get mostly just copied, only the area around the head changes.
- After a terminating configuration has been reached: make row equal by deleting the configuration.

...

Reducibility of H to MPCP(3)

Proof (continued).

Alphabet of C is $\Gamma \cup Q \cup \{\#\}$.

1. Pair: $(\#, \#q_0w\#)$

Other pairs:

1. copy: (a, a) for all $a \in \Gamma \cup \{\#\}$
2. transition:

$$(qa, cq') \text{ if } \delta(q, a) = (q', c, R)$$
$$(q\#, cq'\#) \text{ if } \delta(q, \square) = (q', c, R)$$

...

Reducibility of H to MPCP(4)

Proof (continued).

$$(bqa, q'bc) \text{ if } \delta(q,a) = (q', c, L) \text{ for all } b \in \Gamma$$
$$(bq\#, q'bc\#) \text{ if } \delta(q,\Box) = (q', c, L) \text{ for all } b \in \Gamma$$
$$(\#qa, \#q'c) \text{ if } \delta(q,a) = (q', c, L)$$
$$(\#q\#, \#q'c\#) \text{ if } \delta(q,\Box) = (q', c, L)$$

3. deletion: (aq, q) and (qa, q)
   for all $a \in \Gamma$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$
4. finish: $(q\#\#, \#)$ for all $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$

...

Reducibility of H to MPCP(5)

Proof (continued).

"$\Rightarrow$" If M terminates on input w, there is a sequence $c_0, \dots, c_t$ of configurations with

- $c_0 = q_0 w$ is the start configuration
- c_t is a terminating configuration ($c_t = uqv$ mit $u, v \in \Gamma^*$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$)
- $c_i \vdash c_{i+1}$ for $i = 0, 1, \dots, t-1$

Then C has a match with the overall word

$$\#c_0\#c_1\#\dots\#c_t\#c_t'\#c_t''\#\dots\#q_e\#\#$$

Up to c_t: "'t-row"' follows "'b-row"'

From c_t': deletion of symbols adjacent to terminating state. ...

Reducibility of H to MPCP(6)

Proof (continued).

"$\Leftarrow$" If C has a solution, it has the form

$$\#c_0\#c_1\#\ldots\#c_n\#\#,$$

with $c_0 = q_0 w$. Moreover, there is an $\ell \leq n$, such that q_{accept} or q_{reject} occurs for the first time in c_ℓ.
All c_i for $i \leq \ell$ are configurations of M and $c_i \vdash c_{i+1}$ for $i \in \{0, \ldots, \ell - 1\}$.
$c_0, \ldots, c_\ell$ is hence the sequence of configurations of M on input w, which shows that the TM terminates. $\square$

PCP: Undecidability – Done!

Theorem (Undecidability of PCP)

PCP *is undecidable.*

Proof via an intermediate other problem
modified PCP (MPCP)

1. Reduce MPCP to PCP (MPCP $\leq$ PCP) ✓
2. Reduce halting problem to MPCP ($H \leq$ MPCP) ✓

Proof.

Due to $H \leq$ MPCP and MPCP $\leq$ PCP it holds that $H \leq$ PCP.
Since H is undecidable, also PCP must be undecidable. □

C5.3 Summary

Summary

- Post Correspondence Problem:
  Find a sequence of word pairs s.t. the concatenation of all first components equals the one of all second components.
- The Post Correspondence Problem is Turing-recognizable but not decidable.