Theory of Computer Science

C4. Reductions

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Introduction

What We Achieved So Far: Discussion

- We already know a concrete undecidable problem.
  $\rightarrow$ halting problem
- We will see that we can derive further undecidability results from the undecidability of the halting problem.
- The central notion for this is reducing one problem to another problem.

Illustration

```
def is_odd(some_number):
    n = some_number + 1
    return is_even(n)
```

- Decides whether a given number is odd based on...
- an algorithm that determines whether a number is even.

Reduction: Idea (slido)

Assume that you have an algorithm that solves problem A relying on a hypothetical algorithm for problem B.

```
def is_in_A(input_A):
  input_B = <compute suitable instance based on input_A>
  return is_in_B(input_B)
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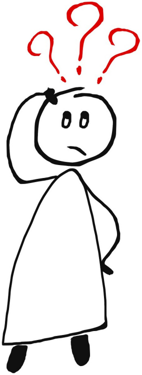
What (if anything) can you conclude

1. if there indeed is an algorithm for problem A?
2. if there indeed is an algorithm for problem B?
3. if problem A is undecidable?
4. if problem B is undecidable?

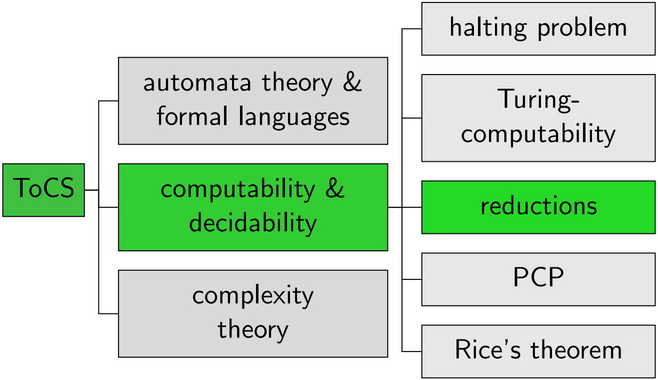
Questions



Questions?

Reduction

Content of the Course



Reduction: Definition

Definition (Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages, and let $f : \Sigma^* \to \Gamma^*$ be a total and computable function such that for all $x \in \Sigma^*$:

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Then we say that A can be reduced to B (in symbols: $A \leq B$), and f is called a reduction from A to B.

German: A ist auf B reduzierbar, Reduktion von A auf B

Reduction Property

Theorem (Reductions vs. Turing-recognizability/Decidability)

Let A and B be languages with $A \leq B$. Then:

1. *If B is decidable, then A is decidable.*
2. *If B is Turing-recognizable, then A is Turing-recognizable.*
3. *If A is not decidable, then B is not decidable.*
4. *If A is not Turing-recognizable, then B is not Turing-recognizable.*

$\rightsquigarrow$ In the following, we use 3. to show undecidability for further problems.

Reduction Property: Proof

Proof.

for 1.: If B is decidable then there is a DTM M_B that decides B. The following algorithm decides A using reduction f from A to B.

On input x:

1. $y := f(x)$
2. Simulate M_B on input y. This simulation terminates.
3. If M_B accepted y, accept. Otherwise reject.

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for 2.: identical to (1), only that M_B only recognizes B and therefore the simulation does not necessarily terminate if $y \notin B$. Since $y \notin B$ iff $x \notin A$, the procedure still recognizes A.

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for 3./4.: contrapositions of 1./2. $\rightsquigarrow$ logically equivalent □

Reductions are Preorders

Theorem (Reductions are Preorders)

The relation "$\leq$" is a preorder:

1. *For all languages A:*
   $A \leq A$ *(reflexivity)*
2. *For all languages A, B, C:*
   If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity)

German: schwache Halbordnung/Quasiordnung, Reflexivität, Transitivität

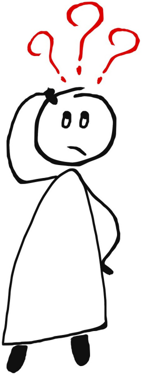
Reductions are Preorders: Proof

Proof.

for 1.: The function $f(x) = x$ is a reduction from A to A because it is total and computable and $x \in A$ iff $f(x) \in A$.

for 2.: $\rightsquigarrow$ exercises $\square$

Questions



Questions?

Halting Problem on Empty Tape

Example

As an example

- we will consider problem H_0, a variant of the halting problem,
- . . . and show that it is undecidable
- . . . reducing H to H_0.

Reminder: Halting Problem

Definition (Halting Problem)

The halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The halting problem on the empty tape is the language

$$H_0 = \{w \in \{0,1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

German: Halteproblem auf leerem Band

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The halting problem on the empty tape is the language

$$H_0 = \{w \in \{0,1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

Note: H_0 is Turing-recognizable. (Why?)

German: Halteproblem auf leerem Band

Halting Problem on Empty Tape (2)

Theorem (Undecidability of Halting Problem on Empty Tape)

The halting problem on the empty tape is undecidable.

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Proof.

We show $H \leq H_0$.

Consider the function $f : \{0, 1, \#\}^* \to \{0, 1\}^*$
that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

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- Test if z has the form $w\#x$ with $w, x \in \{0, 1\}^*$.
- If not, return any word that is not in H_0
  (e. g., encoding of a TM that instantly starts an endless loop).
- If yes, split z into w and x.

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- If yes, split z into w and x.
- Decode w to a TM M_2.

...

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
  - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
  - otherwise, stop immediately

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- Construct TM M that first runs M_1 and then M_2.
  $\rightarrow$ M started on empty tape simulates M_2 on input x.

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f is total and (with some effort) computable. Also:

$$\begin{aligned} z \in H &\text{ iff } z = w\#x \text{ and } M_w \text{ run on } x \text{ terminates} \\ &\text{ iff } M_{f(z)} \text{ started on empty tape terminates} \\ &\text{ iff } f(z) \in H_0 \end{aligned}$$

$\rightsquigarrow H \leq H_0 \rightsquigarrow H_0$ undecidable $\square$

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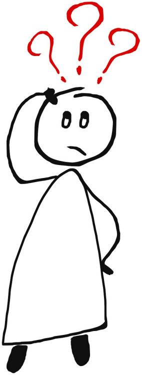

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Summary

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- reductions: "embedding" a problem as a special case of another problem
- important method for proving undecidability: reduce from a known undecidable problem to a new problem