

# Theory of Computer Science

## C4. Reductions

Gabriele Röger

University of Basel

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# C4.1 Introduction

## What We Achieved So Far: Discussion

- ▶ We already know a concrete undecidable problem.  
→ halting problem
- ▶ We will see that we can derive further undecidability results from the undecidability of the halting problem.
- ▶ The central notion for this is reducing one problem to another problem.

# Illustration

```
def is_odd(some_number):  
    n = some_number + 1  
    return is_even(n)
```

- ▶ Decides whether a given number is odd based on...
- ▶ an algorithm that determines whether a number is even.

## Reduction: Idea (slido)

Assume that you have an algorithm that solves problem  $A$  relying on a hypothetical algorithm for problem  $B$ .

```
def is_in_A(input_A):  
    input_B = <compute suitable instance based on input_A>  
    return is_in_B(input_B)
```

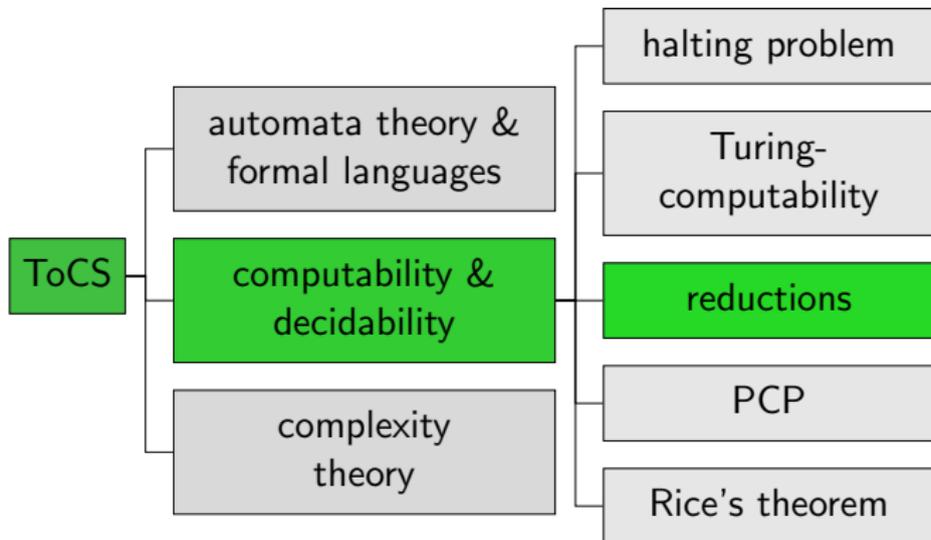
What (if anything) can you conclude

- 1 if there indeed is an algorithm for problem  $A$ ?
- 2 if there indeed is an algorithm for problem  $B$ ?
- 3 if problem  $A$  is undecidable?
- 4 if problem  $B$  is undecidable?



## C4.2 Reduction

# Content of the Course



## Reduction: Definition

### Definition (Reduction)

Let  $A \subseteq \Sigma^*$  and  $B \subseteq \Gamma^*$  be languages, and let  $f : \Sigma^* \rightarrow \Gamma^*$  be a total and computable function such that for all  $x \in \Sigma^*$ :

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Then we say that  $A$  can be **reduced to  $B$**  (in symbols:  $A \leq B$ ), and  $f$  is called a **reduction from  $A$  to  $B$** .

German:  $A$  ist auf  $B$  reduzierbar, Reduktion von  $A$  auf  $B$

# Reduction Property

## Theorem (Reductions vs. Turing-recognizability/Decidability)

Let  $A$  and  $B$  be languages with  $A \leq B$ . Then:

- 1 If  $B$  is decidable, then  $A$  is decidable.
- 2 If  $B$  is Turing-recognizable, then  $A$  is Turing-recognizable.
- 3 If  $A$  is not decidable, then  $B$  is not decidable.
- 4 If  $A$  is not Turing-recognizable, then  $B$  is not Turing-recognizable.

$\rightsquigarrow$  In the following, we use 3. to show undecidability for further problems.

## Reduction Property: Proof

Proof.

for 1.: If  $B$  is decidable then there is a DTM  $M_B$  that decides  $B$ . The following algorithm decides  $A$  using reduction  $f$  from  $A$  to  $B$ .

On input  $x$ :

- 1  $y := f(x)$
- 2 Simulate  $M_B$  on input  $y$ . This simulation terminates.
- 3 If  $M_B$  accepted  $y$ , accept. Otherwise reject.

for 2.: identical to (1), only that  $M_B$  only recognizes  $B$  and therefore the simulation does not necessarily terminate if  $y \notin B$ . Since  $y \notin B$  iff  $x \notin A$ , the procedure still recognizes  $A$ .

for 3./4.: contrapositions of 1./2.  $\rightsquigarrow$  logically equivalent □

# Reductions are Preorders

## Theorem (Reductions are Preorders)

The relation " $\leq$ " is a preorder:

① For all languages  $A$ :

$A \leq A$  (*reflexivity*)

② For all languages  $A, B, C$ :

If  $A \leq B$  and  $B \leq C$ , then  $A \leq C$  (*transitivity*)

German: schwache Halbordnung/Quasiordnung, Reflexivität, Transitivität

## Reductions are Preorders: Proof

Proof.

for 1.: The function  $f(x) = x$  is a reduction from  $A$  to  $A$  because it is total and computable and  $x \in A$  iff  $f(x) \in A$ .

for 2.:  $\rightsquigarrow$  exercises



## C4.3 Halting Problem on Empty Tape

## Example

As an example

- ▶ we will consider problem  $H_0$ , a variant of the halting problem,
- ▶ ...and show that it is undecidable
- ▶ ...reducing  $H$  to  $H_0$ .

## Reminder: Halting Problem

### Definition (Halting Problem)

The **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

# Halting Problem on Empty Tape (1)

## Definition (Halting Problem on the Empty Tape)

The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

**Note:**  $H_0$  is Turing-recognizable. (Why?)

German: Halteproblem auf leerem Band

## Halting Problem on Empty Tape (2)

### Theorem (Undecidability of Halting Problem on Empty Tape)

*The halting problem on the empty tape is undecidable.*

#### Proof.

We show  $H \leq H_0$ .

Consider the function  $f : \{0, 1, \#\}^* \rightarrow \{0, 1\}^*$

that computes the word  $f(z)$  for a given  $z \in \{0, 1, \#\}^*$  as follows:

- ▶ Test if  $z$  has the form  $w\#x$  with  $w, x \in \{0, 1\}^*$ .
- ▶ If not, return any word that is not in  $H_0$   
(e. g., encoding of a TM that instantly starts an endless loop).
- ▶ If yes, split  $z$  into  $w$  and  $x$ .
- ▶ Decode  $w$  to a TM  $M_2$ .

...

## Halting Problem on Empty Tape (3)

Proof (continued).

- ▶ Construct a TM  $M_1$  that behaves as follows:
  - ▶ If the input is empty: write  $x$  onto the tape and move the head to the first symbol of  $x$  (if  $x \neq \varepsilon$ ); then stop
  - ▶ otherwise, stop immediately
- ▶ Construct TM  $M$  that first runs  $M_1$  and then  $M_2$ .  
 →  $M$  started on empty tape simulates  $M_2$  on input  $x$ .
- ▶ Return the encoding of  $M$ .

$f$  is total and (with some effort) computable. Also:

$$\begin{aligned}
 z \in H &\text{ iff } z = w\#x \text{ and } M_w \text{ run on } x \text{ terminates} \\
 &\text{ iff } M_{f(z)} \text{ started on empty tape terminates} \\
 &\text{ iff } f(z) \in H_0
 \end{aligned}$$

$$\rightsquigarrow H \leq H_0 \rightsquigarrow H_0 \text{ undecidable}$$



# Summary

- ▶ **reductions:** “embedding” a problem as a special case of another problem
- ▶ important method for proving undecidability:  
reduce from a known undecidable problem to a new problem