Theory of Computer Science

C3. Turing-Computability

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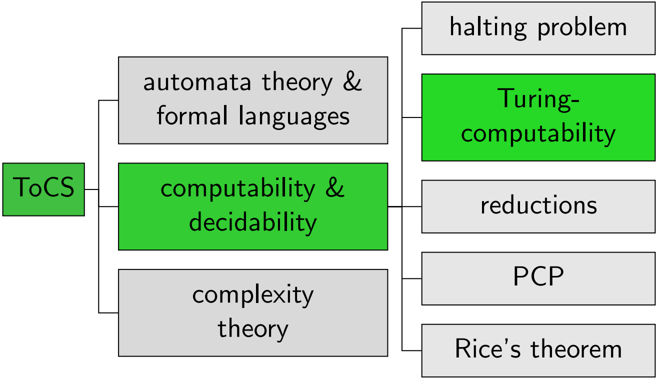
Content of the Course



C3.1 Turing-Computable Functions

Hello World

```
def hello_world(name):
    return "Hello " + name + "!"
```

When calling `hello_world("David")`
we get the result `"Hello David!"`.

How could a Turing machine output a string
as the result of a computation?

Church-Turing Thesis Revisited

Church-Turing Thesis

All functions that can be computed in the intuitive sense can be computed by a Turing machine.

- Talks about arbitrary functions that can be computed in the intutive sense.
- So far, we have only considered recognizability and decidability: Is a word in a language, yes or no?
- We now will consider function values beyond yes or no (accept or reject).
- $\Rightarrow$ consider the tape content when the TM accepted.

Computation

In the following we investigate
models of computation for partial functions $f : \mathbb{N}_0^k \rightarrow_\mathsf{p} \mathbb{N}_0$.

- no real limitation: arbitrary information can be encoded as numbers

German: Berechnungsmodelle

Reminder: Configurations and Computation Steps

How do Turing Machines Work?

- configuration: $\langle \alpha, q, \beta \rangle$ with $\alpha \in \Gamma^*$, $q \in Q$, $\beta \in \Gamma^+$
- one computation step: $c \vdash c'$ if one computation step can turn configuration c into configuration c'
- multiple computation steps: $c \vdash^* c'$ if 0 or more computation steps can turn configuration c into configuration c' ($c = c_0 \vdash c_1 \vdash c_2 \vdash \dots \vdash c_{n-1} \vdash c_n = c'$, $n \geq 0$)

(Definition of $\vdash$, i.e., how a computation step changes the configuration, is not repeated here. ⇝ Chapter B11)

Computation of Functions?

How can a DTM compute a function?

- "Input" x is the initial tape content.
- "Output" $f(x)$ is the tape content (ignoring blanks at the right) when reaching the accept state.
- If the TM stops in the reject state or does not stop for the given input, $f(x)$ is undefined for this input.

Which kinds of functions can be computed this way?

- directly, only functions on words: $f : \Sigma^* \to_p \Sigma^*$
- interpretation as functions on numbers $f : \mathbb{N}_0^k \to_p \mathbb{N}_0$: encode numbers as words

Turing Machines: Computed Function

Definition (Function Computed by a Turing Machine)

A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ **computes** the (partial) function $f : \Sigma^* \to_{\text{p}} \Sigma^*$ for which for all $x, y \in \Sigma^*$:

$$f(x) = y \text{ iff } \langle \varepsilon, q_0, x \rangle \vdash^* \langle \varepsilon, q_{\text{accept}}, y\Box \dots \Box \rangle.$$

(special case: initial configuration $\langle \varepsilon, q_0, \Box \rangle$ if $x = \varepsilon$)

- What happens if the computation does not reach q_{accept}?
- What happens if symbols from $\Gamma \setminus \Sigma$ (e. g., $\Box$) occur in y?
- What happens if the read-write head is not at the first tape cell when accepting?
- Is f uniquely defined by this definition? Why?

German: DTM berechnet f

Turing-Computable Functions on Words

Definition (Turing-Computable, $f : \Sigma^* \to_\mathsf{p} \Sigma^*$)

A (partial) function $f : \Sigma^* \to_\mathsf{p} \Sigma^*$ is called Turing-computable if a DTM that computes f exists.

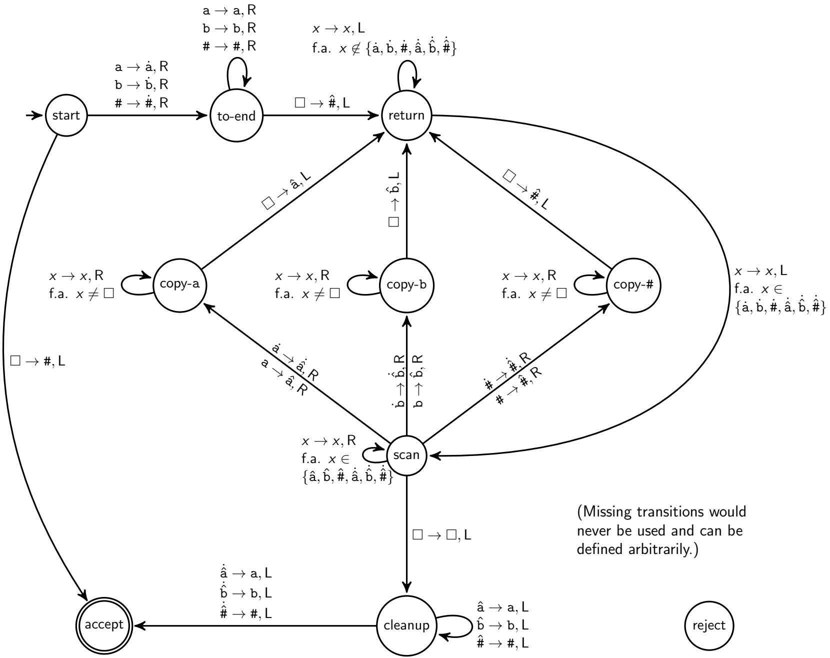
German: Turing-berechenbar

Example: Turing-Computable Functions on Words

Example

Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{\#}\}$.
The function $f : \Sigma^* \rightarrow_{\mathrm{p}} \Sigma^*$ with $f(w) = w\mathtt{\#}w$ for all $w \in \Sigma^*$ is Turing-computable.

Idea: $\rightsquigarrow$ blackboard



(Missing transitions would never be used and can be defined arbitrarily.)

Turing-Computable Numerical Functions

- We now transfer the concept to partial functions $f : \mathbb{N}_0^k \to_{\mathsf{p}} \mathbb{N}_0$.
- Idea:
  - To represent a number as a word, we use its binary representation (= a word over $\{0, 1\}$).
  - To represent tuples of numbers, we separate the binary representations with symbol #.
- For example: $(5, 2, 3)$ becomes 101#10#11

Encoding Numbers as Words

Definition (Encoded Function)

Let $f : \mathbb{N}_0^k \rightarrow_p \mathbb{N}_0$ be a (partial) function.
The encoded function f^{code} of f is the partial function $f^{\text{code}} : \Sigma^* \rightarrow_p \Sigma^*$ with $\Sigma = \{0, 1, \#\}$ and $f^{\text{code}}(w) = w'$ iff

- there are $n_1, \dots, n_k, n' \in \mathbb{N}_0$ such that
- $f(n_1, \dots, n_k) = n'$,
- $w = bin(n_1)\#\dots\#bin(n_k)$ and
- $w' = bin(n')$.

Here $bin : \mathbb{N}_0 \rightarrow \{0, 1\}^*$ is the binary encoding (e. g., $bin(5) = 101$).

Example: $f(5, 2, 3) = 4$ corresponds to $f^{\text{code}}(101\#10\#11) = 100$.

German: codierte Funktion

Turing-Computable Numerical Functions

Definition (Turing-Computable, $f : \mathbb{N}_0^k \to_{\mathrm{p}} \mathbb{N}_0$)

A (partial) function $f : \mathbb{N}_0^k \to_{\mathrm{p}} \mathbb{N}_0$ is called Turing-computable if a DTM that computes f^{code} exists.

German: Turing-berechenbar

Exercise

The addition of natural numbers $+ : \mathbb{N}_0^2 \to \mathbb{N}_0$ is Turing-computable. You have a TM M that computes $+^{\text{code}}$.

You want to use M to compute the sum $3 + 2$.
What is your input to M?

Example: Turing-Computable Numerical Function

Example

The following numerical functions are Turing-computable:

- $succ : \mathbb{N}_0 \rightarrow_{\mathsf{p}} \mathbb{N}_0$ with $succ(n) := n + 1$
- $pred_1 : \mathbb{N}_0 \rightarrow_{\mathsf{p}} \mathbb{N}_0$ with $pred_1(n) := \begin{cases} n - 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$
- $pred_2 : \mathbb{N}_0 \rightarrow_{\mathsf{p}} \mathbb{N}_0$ with $pred_2(n) := \begin{cases} n - 1 & \text{if } n \geq 1 \\ \text{undefined} & \text{if } n = 0 \end{cases}$

How does incrementing and decrementing binary numbers work?

Successor Function

The Turing machine for *succ* works as follows:

(Details of marking the first tape position ommitted)

1. Check that the input is a valid binary number:
   - If the input is not a single symbol 0 but starts with a 0, reject.
   - If the input contains symbol #, reject.
2. Move the head onto the last symbol of the input.
3. While you read a 1 and you are not at the first tape position, replace it with a 0 and move the head one step to the left.
4. Depending on why the loop in stage 3 terminated:
   - If you read a 0, replace it with a 1, move the head to the left end of the tape and accept.
   - If you read a 1 at the first tape position, move every non-blank symbol on the tape one position to the right, write a 1 in the first tape position and accept.

Predecessor Function

The Turing machine for $pred_1$ works as follows:

(Details of marking the first tape position ommitted)

1. Check that the input is a valid binary number (as for *succ*).
2. If the (entire) input is 0 or 1, write a 0 and accept.
3. Move the head onto the last symbol of the input.
4. While you read symbol 0 replace it with 1 and move left.
5. Replace the 1 with a 0.
6. If you are on the first tape cell, eliminate the trailing 0 (moving all other non-blank symbols one position to the left).
7. Move the head to the first position and accept.

What do you have to change to get a TM for $pred_2$?

More Turing-Computable Numerical Functions

Example

The following numerical functions are Turing-computable:

- $add : \mathbb{N}_0^2 \to_p \mathbb{N}_0$ with $add(n_1, n_2) := n_1 + n_2$
- $sub : \mathbb{N}_0^2 \to_p \mathbb{N}_0$ with $sub(n_1, n_2) := \max\{n_1 - n_2, 0\}$
- $mul : \mathbb{N}_0^2 \to_p \mathbb{N}_0$ with $mul(n_1, n_2) := n_1 \cdot n_2$
- $div : \mathbb{N}_0^2 \to_p \mathbb{N}_0$ with $div(n_1, n_2) := \begin{cases} \left\lceil \frac{n_1}{n_2} \right\rceil & \text{if } n_2 \neq 0 \\ \text{undefined} & \text{if } n_2 = 0 \end{cases}$

$\rightsquigarrow$ sketch?

C3.2 Decidability vs. Computability

Decidability as Computability

Theorem

A language $L \subseteq \Sigma^*$ is decidable iff $\chi_L : \Sigma^* \to \{0, 1\}$, the characteristic function of L, is computable.

Here, for all $w \in \Sigma^*$:

$$\chi_L(w) := \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{if } w \notin L \end{cases}$$

Proof sketch.

"$\Rightarrow$" Let M be a DTM for L. Construct a DTM M' that simulates M on the input. If M accepts, M' writes a 1 on the tape. If M rejects, M' writes a 0 on the tape. Afterwards M' accepts.

"$\Leftarrow$" Let C be a DTM that computes χ_L. Construct a DTM C' that simulates C on the input. If the output of C is 1 then C' accepts, otherwise it rejects.

Turing-recognizable Languages and Computability

Theorem

A language $L \subseteq \Sigma^*$ is Turing-recognizable iff the following function $\chi'_L : \Sigma^* \to_p \{0, 1\}$ is computable.

Here, for all $w \in \Sigma^*$:

$$\chi'_L(w) = \begin{cases} 1 & \text{if } w \in L \\ \text{undefined} & \text{if } w \notin L \end{cases}$$

Proof sketch.

"$\Rightarrow$" Let M be a DTM for L. Construct a DTM M' that simulates M on the input. If M accepts, M' writes a 1 on the tape and accepts. Otherwise it enters an infinite loop.

"$\Leftarrow$" Let C be a DTM that computes χ'_L. Construct a DTM C' that simulates C on the input. If C accepts with output 1 then C' accepts, otherwise it enters an infinite loop.

C3.3 Summary

Summary

- Turing-computable function $f : \Sigma^* \rightarrow_{\text{p}} \Sigma^*$:
  there is a DTM that transforms every input $w \in \Sigma^*$ into the output $f(w)$ (undefined if DTM does not stop or stops in invalid configuration)
- Turing-computable function $f : \mathbb{N}_0^k \rightarrow_{\text{p}} \mathbb{N}_0$:
  ditto; numbers encoded in binary and separated by #