# Theory of Computer Science

## C2. The Halting Problem

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# C2.1 Turing-recognizable vs. decidable





## Plan for this Chapter

- We will first revisit the notions Turing-recognizable and Turing-decidable and identify a connection between the two concepts.
- Then we will get to know an important undecidable problem, the halting problem.
- We show that it is Turing-recognizable. . .
- . . . but not Turing-decidable.
- From these results we can conclude that there are languages that are not Turing-recognizable.
- Some of the postponed results on the closure and decidability properties of type 0 languages are direct implications of our findings.

## Reminder: Turing-recognizable and Turing-decidable

**Definition (Turing-recognizable Language)**

We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

A Turing machine that halts on all inputs (entering $q_{\text{reject}}$ or $q_{\text{accept}}$) is a decider. A decider that recognizes some language also is said to decide the language.

**Definition (Turing-decidable Language)**

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

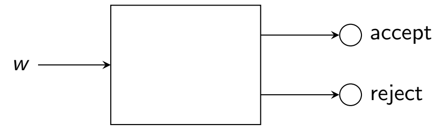




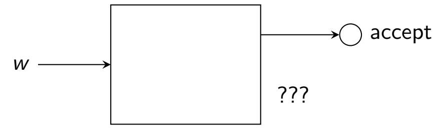
## Intuition

Are these two definitions meaningfully different? Yes!

(Turing-)decidable:




Turing-recognizable








## Connection Turing-recognizable and Turing-decidable (1)

Reminder: For language $L$, we write $\bar{L}$ do denote its complement.

**Theorem (Decidable vs. Turing-recognizable)**

*A language $L$ is decidable iff both $L$ and $\bar{L}$ are Turing-recognizable.*

**Proof.**

($\Rightarrow$): obvious (Why?) ...





## Connection Turing-recognizable and Turing-decidable (2)

**Proof (continued).**

($\Leftarrow$): Let $M_L$ be a DTM that recognizes $L$,
and let $M_{\bar{L}}$ be a DTM that recognizes $\bar{L}$.

The following algorithm decides $L$:

On a given input word $w$ proceed as follows:
FOR $s := 1, 2, 3, \dots$:
  IF $M_L$ stops on $w$ in $s$ steps in the accept state:
    ACCEPT
  IF $M_{\bar{L}}$ stops on $w$ in $s$ steps in the accept state:
    REJECT

□

Why don't we first entirely simulate $M_L$ on the input and only afterwards $M_{\bar{L}}$?

## Example: Decidable $\neq$ Known Algorithm

Decidability of $L$ does not mean we know how to decide it:

- $L = \{n \in \mathbb{N} \mid$ there are $n$ consecutive 7s in the decimal representation of $\pi\}$.
- $L$ is decidable.
- There are either 7-sequences of arbitrary length in $\pi$ (case 1) or there is a maximal number $n_0$ of consecutive 7s (case 2).
  - Case 1: accept for all $n$
  - Case 2: accept if $n \leq n_0$, otherwise reject
- In both cases, we can decide the language.
- We just do not know what is the correct version (and what is $n_0$ in case 2).



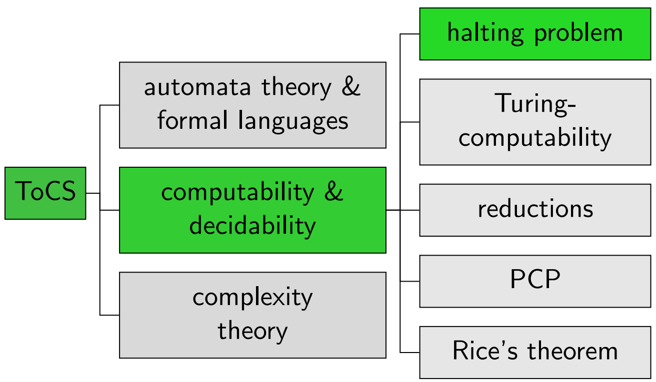


# C2.2 The Halting Problem $H$





## Content of the Course








## Reminder: Encodings of Turing Machines

- We have seen how every deterministic Turing machine with input alphabet $\{0, 1\}$ can be encoded as a word over $\{0, 1\}$. Can there be several words that encode the same DTM?
- Not every word over $\{0, 1\}$ corresponds to such an encoding.
- To define for every $w \in \{0, 1\}^*$ a corresponding TM, we use an arbitrary fixed DTM $\widehat{M}$ and define

$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \widehat{M} & \text{otherwise} \end{cases}$$

- $M_w =$ "Turing machine encoded by $w$"

## Halting Problem

**Definition (Halting Problem)**
The halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

"Does the computation of the TM encoded by $w$ halt on input $x$?"
"Does a given piece of code terminate on a given input?"





## The Halting Problem is Turing-recognizable

**Theorem**
*The halting problem $H$ is Turing-recognizable.*

The following Turing machine $U$ recognizes language $H$:

On input $w\#x$:

1. If the input contains more than one $\#$ then reject.
2. Simulate $M_w$ (the TM encoded by $w$) on input $x$.
3. If $M_w$ halts, accept.

What does $U$ do if $M_w$ does not halt on the input?

$U$ is an example of a so-called *universal Turing machine* which can simulate any other Turing machine from the description of that machine.





# C2.3 $H$ is Undecidable





## Undecidability

- If some language or problem is not Turing-decidable then we call it undecidable.
- Intuitively, this means that for this problem there is no algorithm that is correct and terminates on all inputs.
- To establish the undeciability of the halting problem, we will consider a situation where we run a Turing machine/algorithm on its own encoding/source code.
- We have seen something similar in the very first lecture...

## Uncomputable Problems?

Consider functions whose inputs are strings:

```
def program_returns_true_on_input(prog_code, input_str):
    ...
    # returns True if prog_code run on input_str returns True
    # returns False if not

def weird_program(prog_code):
    if program_returns_true_on_input(prog_code, prog_code):
        return False
    else:
        return True
```

What is the return value of `weird_program` if we run it on its own source code?





## Solution

- We can make a case distinction:
  - Case 1: `weird_program` returns True on its own source. Then `weird_program` returns False on its own source code.
  - Case 2: `weird_program` returns False on its own source. Then `weird_program` returns True on its own source code.
- Contradiction in all cases, so `weird_program` cannot exist.
- From the source we see that this can only be because subroutine `program_returns_true_on_input` cannot exist.
- Overall, we have proven that there cannot be a program with the behaviour described by the comments.
- For the undecidability of the halting problem, we will use an analogous argument, only with Turing machines instead of code and termination instead of return values.





## Undecidability of the Halting Problem (1)

**Theorem (Undecidability of the Halting Problem)**

*The halting problem $H$ is undecidable.*

**Proof.**

Proof by contradiction: we assume that the halting problem $H$ was decidable and derive a contradiction.

So assume $H$ is decidable and let $D$ be a DTM that decides it. ...





## Undecidability of the Halting Problem (2)

**Proof (continued).**

Construct the following new machine $M$ that takes a word $x \in \{0,1\}^*$ as input:

1. Execute $D$ on the input $x\#x$.
2. If it rejects: accept.
3. Otherwise: enter an endless loop.

Let $w$ be the encoding of $M$. How will $M$ behave on input $w$?

$M$ run on $w$ stops
iff $D$ run on $w\#w$ rejects
iff $w\#w \notin H$
iff $M$ run on $w$ does not stop (remember that $w$ encodes $M$)

Contradiction! DTM $M$ cannot exist.
$\Rightarrow$ DTM $D$ cannot exist, thus $H$ is not decidable. $\square$

## A Language that is not Turing-recognizable

We have the following results:

- A language $L$ is decidable iff both $L$ and $\bar{L}$ are Turing-recognizable.
- The halting problem $H$ is Turing-recognizable but not decidable.

**Corollary**

*The complement $\bar{H}$ of the halting problem $H$ is not Turing-recognizable.*





## Exercises

- True or false? There is a grammar that generates $H$.
- True or false? Not all languages are of type 0.

Justify your answers.





# C2.4 Reprise: Type-0 Languages





## Back to Chapter B13: Closure Properties

| | Intersection | Union | Complement | Concatenation | Star |
|---|---|---|---|---|---|
| Type 3 | Yes | Yes | Yes | Yes | Yes |
| Type 2 | No | Yes | No | Yes | Yes |
| Type 1 | Yes$^{(2)}$ | Yes$^{(1)}$ | Yes$^{(2)}$ | Yes$^{(1)}$ | Yes$^{(1)}$ |
| Type 0 | Yes$^{(2)}$ | Yes$^{(1)}$ | No$^{(3)}$ | Yes$^{(1)}$ | Yes$^{(1)}$ |

Proofs?
(1) proof via grammars, similar to context-free cases
(2) without proof
(3) proof in later chapters (part C)

## Back to Chapter B13: Decidability

| | Word problem | Emptiness problem | Equivalence problem | Intersection problem |
|---|---|---|---|---|
| Type 3 | Yes | Yes | Yes | Yes |
| Type 2 | Yes | Yes | No | No |
| Type 1 | Yes$^{(1)}$ | No$^{(3)}$ | No$^{(2)}$ | No$^{(2)}$ |
| Type 0 | No$^{(4)}$ | No$^{(4)}$ | No$^{(4)}$ | No$^{(4)}$ |

Proofs?
(1) same argument we used for context-free languages
(2) because already undecidable for context-free languages
(3) without proof
(4) proofs in later chapters (part C)





## Answers to Old Questions

Closure properties:

- $H$ is Turing-recognizable (and thus type 0) but not decidable.
- ⇝ $\bar{H}$ is not Turing-recognizable, thus not type 0.
- ⇝ Type-0 languages are not closed under complement.

Decidability:

- $H$ is type 0 but not decidable.
- ⇝ word problem for type-0 languages not decidable
- ⇝ emptiness, equivalence, intersection problem: later in exercises
  (We are still missing some important results for this.)





# C2.5 Summary





## Summary

- A language $L$ is decidable iff both $L$ and $\bar{L}$ are Turing-recognizable.
- The halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

- The halting problem is Turing-recognizable but undecidable.
- The complement language $\bar{H}$ is an example of a language that is not even Turing-recognizable.