

# Theory of Computer Science

## C1. Turing Machines as Formal Model of Computation

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# Overview: Course

contents of this course:

A. background ✓

- ▷ mathematical foundations and proof techniques

B. automata theory and formal languages ✓

- ▷ What is a computation?

C. Turing computability

- ▷ What can be computed at all?

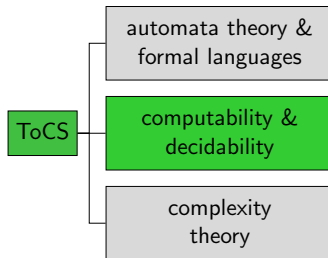
D. complexity theory

- ▷ What can be computed efficiently?

E. more computability theory

- ▷ Other models of computability

# Content of the Course



# Main Question

Main question in this part of the course:

What can be computed  
by a computer?

# Hilbert's 10th Problem

# Algorithms

- Informally, an **algorithm** is a collection of simple instructions for carrying out some task.
- Long history in mathematics since ancient times: descriptions of algorithms e. g. for finding prime numbers or the greatest common divisor.
- A formal notion of an algorithm itself was not defined until the 20th century.

# Hilbert's 10th Problem

Around 1900 David Hilbert (German mathematician) formulated 23 mathematical problems as challenge for the 20th century.

## Hilbert's 10th problem

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients:  
To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

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What does this mean?



# Diophantine Equations

- A **polynomial** is a sum of terms where each term is a product of a constant (the **coefficient**) and certain **variables**.  
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- A **polynomial equation** is an equation  $p = 0$ , where  $p$  is a polynomial. A solutions of the equation is called a **root** of  $p$ .  
e. g.  $6x^3yz^2 + 3xy^2 - x^3 - 10$  has a root  $x = 5, y = 3, z = 0$ .

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e. g.  $6x^3yz^2 + 3xy^2 - x^3 - 10$  has a root  $x = 5, y = 3, z = 0$ .
- **Diophantine equations** are polynomial equations, where only **integral roots** (assigning only integer values to the variables) count as solutions.

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Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients:  
To **devise a process according to which it can be determined in a finite number of operations** whether the equation is solvable in rational integers.

- ☞ **Specify an algorithm** that takes a polynomial with integer coefficients as input and outputs whether it has an integral root.

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To **devise a process according to which it can be determined in a finite number of operations** whether the equation is solvable in rational integers.

👉 **Specify an algorithm** that takes a polynomial with integer coefficients as input and outputs whether it has an integral root.

**There is no such algorithm!**  
(implication of Matiyasevich's theorem from 1970)

# Questions



Questions?

# Church-Turing Thesis

# Formal Notion of Algorithm?

- What is an algorithm?
  - intuitive model of algorithm (cookbook recipe)
  - vs. algorithm in modern programming language
  - vs. formal mathematical models
- Proving that no algorithm exists requires a clear notion of algorithm.



# Church-Turing Thesis

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All functions that can be **computed in the intuitive sense** can be computed by a **Turing machine**.

- cannot be proven (**why not?**)

German: Church-Turing-These

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  - Counter machine: concept of registers
  - Random-access machine (RAM): adds indirect addressing
  - Random-access stored-program machines: related to the von Neumann architecture (very close to modern computer systems)

German: Church-Turing-These

# What about the Infinite Tape?

- Turing Machines have access to **infinite storage**.
- Computer systems **do not**.
- **However:** A **halting** (in particular: accepting) computation of a TM can only use a **finite** part of the tape.
- If a problem is undecidable, we cannot solve it with a computer, no matter how much memory we provide.

# Turing Completeness

## Church-Turing Thesis

All functions that can be **computed in the intuitive sense** can be computed by a **Turing machine**.

Vice versa:

We say that a programming language is **Turing-complete** to express that it can compute everything a Turing machine can.

- We can show Turing completeness by showing that with the programming language we can simulate any Turing machine.

# Back to Hilbert's Problem

The corresponding formal problem (= language) is

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Formal way to say that “there is no algorithm for this problem”:

*D* is not Turing-decidable.

# Encoding

# Finite Structures as Strings

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  - We will have a look at two examples:
    - Example 1: Encoding of pairs of numbers
    - Example 2: Encoding of Turing machines

# Encoding and Decoding: Binary Encode

Consider the function  $encode : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  with:

$$encode(x, y) := \binom{x + y + 1}{2} + x$$

- $encode$  is known as the **Cantor pairing function**
- $encode$  is computable
- $encode$  is **bijective**

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$y = 0$	0	2	5	9	14
$y = 1$	1	4	8	13	19
$y = 2$	3	7	12	18	25
$y = 3$	6	11	17	24	32
$y = 4$	10	16	23	31	40

German: Cantorsche Paarungsfunktion

# Encoding and Decoding: Binary Decode

Consider the **inverse functions**

$decode_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  and  $decode_2 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  of *encode*:

$$decode_1(encode(x, y)) = x$$

$$decode_2(encode(x, y)) = y$$

- $decode_1$  and  $decode_2$  are computable

# Turing Machines as Inputs

- We will at some point consider problems that have Turing machines as their **input**.  
     $\rightsquigarrow$  “programs that have programs as input”:  
    cf. compilers, interpreters, virtual machines, etc.
- We have to think about how we can encode **arbitrary Turing machines** as **words over a fixed alphabet**.
- We use the binary alphabet  $\Sigma = \{0, 1\}$ .
- As an intermediate step we first encode over the alphabet  $\Sigma' = \{0, 1, \#\}$ .

# Encoding a Turing Machine as a Word (1)

**Step 1:** encode a Turing machine as a word over  $\{0, 1, \#\}$

**Reminder:** Turing machine  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$

**Idea:**

- input alphabet  $\Sigma$  should always be  $\{0, 1\}$
  - enumerate states in  $Q$  and symbols in  $\Gamma$  and consider them as numbers  $0, 1, 2, \dots$
  - blank symbol always receives number 2
  - start state always receives number 0, accept state number 1 and reject state number 2
- (we can special-case machines where the start state is the accept or reject state)

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Then it is sufficient to **only encode  $\delta$**  explicitly:

- $Q$ : all states mentioned in the encoding of  $\delta$
- $\Gamma = \{0, 1, \square, a_3, a_4, \dots, a_k\}$ , where  $k$  is the largest symbol number mentioned in the  $\delta$ -rules

## Encoding a Turing Machine as a Word (2)

encode the rules:

- Let  $\delta(q_i, a_j) = \langle q_{i'}, a_{j'}, D \rangle$  be a rule in  $\delta$ , where the indices  $i, i', j, j'$  correspond to the enumeration of states/symbols and  $D \in \{L, R\}$ .
- encode this rule as
$$w_{i,j,i',j',D} = \#\#\text{bin}(i)\#\text{bin}(j)\#\text{bin}(i')\#\text{bin}(j')\#\text{bin}(m),$$
where  $m = \begin{cases} 0 & \text{if } D = L \\ 1 & \text{if } D = R \end{cases}$
- For every rule in  $\delta$ , we obtain one such word.
- All of these words in sequence (in arbitrary order) encode the Turing machine.



# Encoding a Turing Machine as a Word (3)

Step 2: transform into word over  $\{0, 1\}$  with mapping

$0 \mapsto 00$

$1 \mapsto 01$

$\# \mapsto 11$

Turing machine can be reconstructed from its encoding.

How?

# Encoding a Turing Machine as a Word (4)

## Example (step 1)

$\delta(q_0, a_3) = \langle q_3, a_2, R \rangle$  becomes ##0#11#11#10#1

$\delta(q_3, a_1) = \langle q_1, a_0, L \rangle$  becomes ##11#1#1#0#0

## Example (step 2)

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## Exercise: Encoding of TMs (slido)

What would be the encoding of a transition  
 $\delta(q_0, a_0) = (q_1, a_2, L)$  as word over  $\{0, 1\}$ ?



# Turing Machine Encoded by a Word

**goal:** function that maps any word in  $\{0, 1\}^*$  to a Turing machine

**problem:** not all words in  $\{0, 1\}^*$  are encodings of a Turing machine

**solution:** Let  $\hat{M}$  be an arbitrary fixed deterministic Turing machine (for example one that always immediately stops). Then:

## Definition (Turing Machine Encoded by a Word)

For all  $w \in \{0, 1\}^*$ :

$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \hat{M} & \text{otherwise} \end{cases}$$

# Notation for Encoding

- Most of the time, we will not consider a particular encoding of non-string objects.
- For a single object  $O$ , we will just write  $\langle\langle O \rangle\rangle$  to denote some suitable encoding of  $O$  as a string.
- For several objects  $O_1, \dots, O_n$ , we write  $\langle\langle O_1, \dots, O_n \rangle\rangle$  for their encoding into a single string.
- In the high-level description of a TM we can refer to them as the objects they are because on the lower level the TM can be programmed to handle the encoded representation accordingly.

# Example

$$L = \{\langle\langle G \rangle\rangle \mid G \text{ is a connected undirected graph}\}$$



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We describe a TM that recognizes  $L$ :

On input  $\langle\langle G \rangle\rangle$ , the encoding of a undirected graph  $G$ :

- ① Select the first node of  $G$  and mark it.
- ② Repeat until no more nodes are marked:  
For each node in  $G$ , mark it if it is adjacent to a node that is already marked.
- ③ Scan all the nodes of  $G$  to determine whether they are all marked. If yes, accept, otherwise reject.

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- ③ Scan all the nodes of  $G$  to determine whether they are all marked. If yes, accept, otherwise reject.

Implicit (lower-level detail): If the input does not encode an undirected graph, directly reject.

# Questions



Questions?

# Summary

# Summary

- main question: **what can a computer compute?**
- approach: investigate **formal models of computation**  
→ deterministic Turing machines
- Based on the (existing evidence for the) Church-Turing thesis, we will describe the behaviour of Turing machines on a higher abstraction level (such as pseudo-code).
- The formal restriction of TMs to strings is not a practical limitation but can be handled with suitable encodings.