

# Theory of Computer Science

## B12. Turing Machines II

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## B12.1 Variants of Turing Machines

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B12. Turing Machines II

Variants of Turing Machines

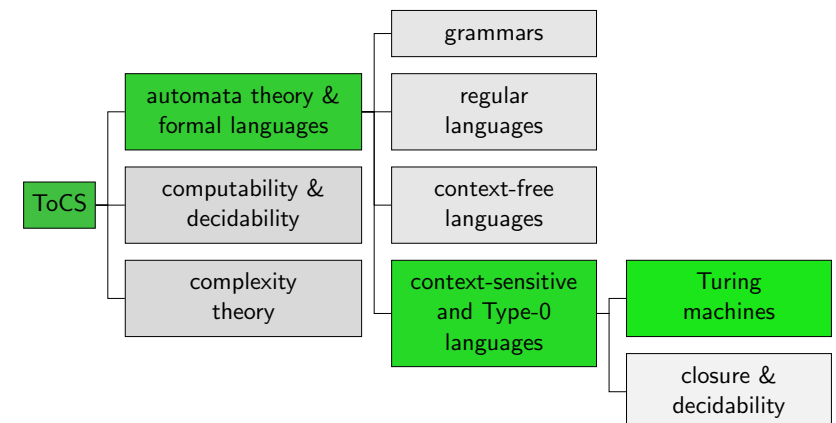
## B12.1 Variants of Turing Machines

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B12. Turing Machines II

Variants of Turing Machines

## Content of the Course



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## Reminder: Deterministic Turing Machine

### Definition (Deterministic Turing Machine)

A (deterministic) **Turing machine (DTM)** is given by a 7-tuple  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ , where  $Q, \Sigma, \Gamma$  are finite and

- ▶  $Q$  is the set of **states**,
- ▶  $\Sigma$  is the **input alphabet**, not containing the **blank symbol**  $\square$ ,
- ▶  $\Gamma$  is the **tape alphabet**, where  $\square \in \Gamma$  and  $\Sigma \subseteq \Gamma$ ,
- ▶  $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$  is the **transition function**,
- ▶  $q_0 \in Q$  is the **start state**,
- ▶  $q_{\text{accept}} \in Q$  is the **accept state**,
- ▶  $q_{\text{reject}} \in Q$  is the **reject state**, where  $q_{\text{accept}} \neq q_{\text{reject}}$ .

**Deterministic** TM with a **single** tape that is infinite at **one side**.

## Turing Machines with Neutral Move

- ▶ A DTM only allows head movements to the **left** or **right**.
- ▶ A DTM **with neutral move** is a variant with transition function  $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, \mathbf{N}\}$ , where **N** means that the R/W-head **stays put**.

Can this variant recognize languages that standard DTMs cannot?

## Turing Machines with Neutral Move are Equally Powerful

- ▶ Obviously, every DTM can be seen as a DTM with neutral move, so the variant is clearly **not less powerful**.
- ▶ Vice versa, every language recognized by a DTM  $M$  with neutral moves can be recognized by a (standard) DTM  $M'$ :
  - ▶ For every state  $q$  of  $M$ ,  $M'$  has an additional state  $q'$ .
  - ▶ If  $M$  writes  $c$ , switches to state  $q$  and stays put,  $M'$  writes  $c$ , switches to state  $q'$  and moves right. Whenever  $M'$  is in one of the primed states, it does not change the tape, switches to the unprimed state  $q$  and moves left.
  - ▶ For every transition of  $M$  with neutral move,  $M'$  takes two transitions with the same result.

To show that two models are equivalent,  
we can show that we can simulate one by the other.

## B12.2 Multitape Turing Machines

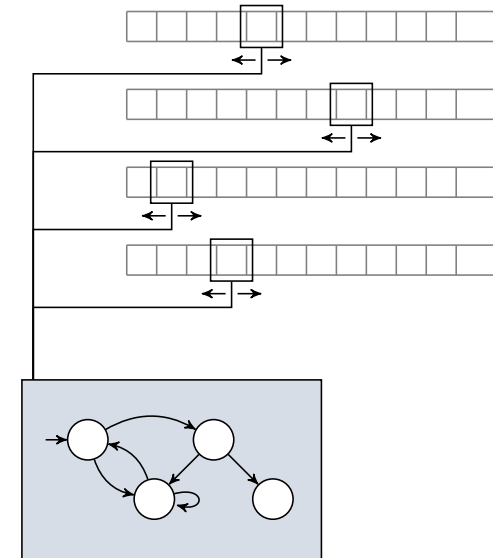
## Multitape Turing Machines

A multitape TM is like a DTM (with neutral movement) but with several tapes.

- ▶ every tape has its own read-write head, initially placed on the left-most cell,
- ▶ the input appears on tape 1,
- ▶ all other tapes are initially filled with blank symbols,
- ▶ the transition function considers all  $k$  tapes simultaneously  
 $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, N\}^k$

German: Mehrband-Turingmaschine

## Multitape Turing Machines: Conceptually



## Multitape Turing Machine: Transitions

$$\delta(q, a_1, \dots, a_k) = (q', a'_1, \dots, a'_k, D_1, \dots, D_k)$$

- ▶ If the TM is in state  $q$ ,
- ▶ and on each tape  $i$  the head reads symbol  $a_i$ , then
- ▶ the TM switches to state  $q'$ ,
- ▶ replaces on each tape  $i$  the symbol  $a_i$  with  $a'_i$ , and
- ▶ moves the head on each tape  $i$  in direction  $D_i$ ;  
 $(D_i \in \{L, R, N\})$

## Multitape TMs No More Powerful Than Single-Tape TMs

### Theorem

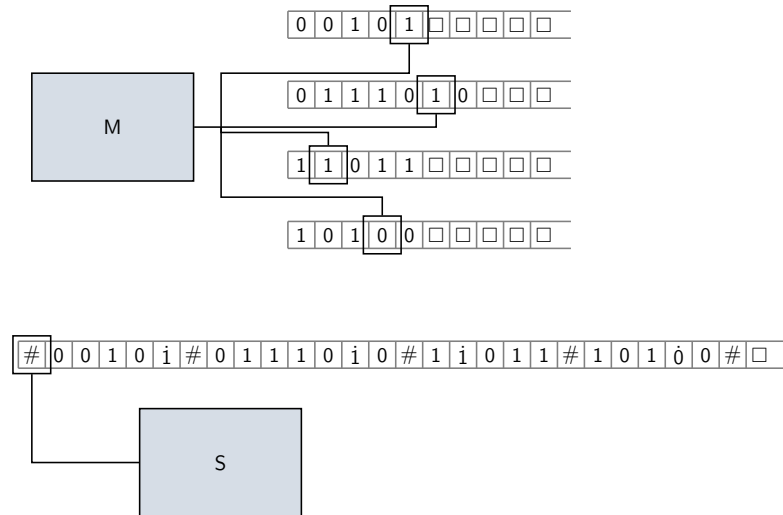
*Every multitape TM has an equivalent single-tape TM.*

### Proof.

Let  $M$  be a TM with  $k$  tapes. We construct a single-tape DTM  $S$  that recognizes the same language.

$S$  stores the information of the multiple tapes on its tape, separating the contents of different tapes with a new symbol  $\#$ .

To keep track of the positions of the heads of  $M$ , TM  $S$  has for each tape symbol  $x$  of  $M$  a new tape symbol  $\dot{x}$  to mark the corresponding positions. ...



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## Multitape TMs No More Powerful Than Single-Tape TMs

### Theorem

*Every multitape TM has an equivalent single-tape TM.*

### Proof (continued).

On input  $w = w_1 \dots w_n$

- ① Initialize the tape of  $S$  to  $\# \dot{w}_1 w_2 \dots w_n \# \dot{\square} \# \dot{\square} \# \dots \#$
- ② To simulate a transition of  $M$ , TM  $S$  scans from the leftmost  $\#$  to the  $k + 1$ st  $\#$  to determine what symbols are under the virtual heads. In a second pass,  $S$  updates the tape according to the transition of  $M$ .
- ③ If it moves a virtual head on the  $\#$  marking the right end of its tape, it frees this position by shifting the tape content from this position on one position to the right and adds a blank into the “new” position.

□

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## Details?

Consider the situation where  $S$  has done its first pass (back at the left-most position) and has determined that  $M$  would take transition

$$\delta(q, x_1, \dots, x_k) = (q, y_1, \dots, y_k, D_1, \dots, D_k).$$

How can you “implement” the second pass of  $S$  that updates the tape accordingly? You may assume that it will never move a virtual head from the already represented part of its tape.



First pass and shifting the tape content  $\rightsquigarrow$  exercises

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## Multitape TMs Equally Powerful as Single-Tape TMs

### Theorem

*A language is Turing-recognizable iff some multitape Turing machine recognizes it.*

### Proof.

“ $\Rightarrow$ ”: A DTM is a special case of a multitape TM.

“ $\Leftarrow$ ”: Previous theorem

□

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## B12.3 Nondeterministic Turing Machines

## Nondeterministic Turing Machines

A nondeterministic Turing machine (NTM) relates to a DTM as a NFA relates to a DFA.

- ▶ The transition function can specify several possibilities:  
 $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R, N\})$
- ▶ For a given input, we can consider the computation tree whose branches correspond to following different possibilities.
- ▶ If some branch leads to the accept state, the NTM accepts the input word.

German: Nichtdeterministische Turingmaschine

## Nondeterministic TMs no More Powerful than DTMs

### Theorem

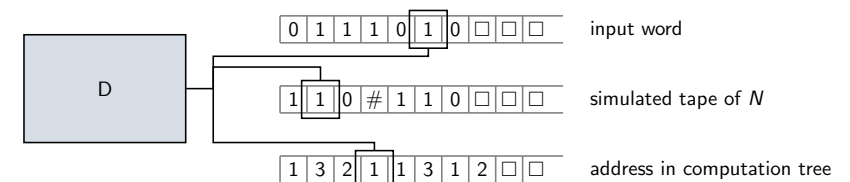
*Every nondeterministic Turing machine has an equivalent deterministic Turing machine.*

## Nondeterministic TMs no More Powerful than DTMs

### Proof.

Let  $N$  be a NTM. We describe a deterministic 3-tape TM  $D$  that searches the computation tree of  $N$  on input  $w$  for an accepting configuration with a breadth-first search. The theorem follows from the equivalence of multitape TMs and DTMs.

The first tape always contains  $w$ , the second tape corresponds to the content of  $N$ 's tape on some branch of the computation tree and the third tape tracks the position in  $N$ 's computation tree. ...



## Nondeterministic TMs no More Powerful than DTMs

What is the “address in the computation tree”?

- ▶ Let  $b$  be the maximal number of children of a node in the CT (= size of largest set of possibilities in the transition function)
- ▶ The address is a string over  $\{1, 2, \dots, b\}$ .
- ▶ For example, address 312 refers to the node in the CT reached by starting from the root node (= initial configuration)
  - ▶ going to the third child node, then
  - ▶ going to the first child of the resulting node, and then
  - ▶ going to the second child of this child node.
- ▶ If a node does not have that many children, the address is *invalid*.

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## Nondeterministic TMs no More Powerful than DTMs

Proof (continued).

$D$  works on input  $w$  as follows:

- ① Initially, tape 1 contains  $w$ , tape 2 and 3 contain only blanks.
- ② Copy tape 1 to tape 2.
- ③ Simulate  $N$  on input  $w$  following one branch of the computation tree. Before each transition of  $N$ , determine which choice to make from the next symbol on tape 3. If there is no number left on tape 3, if the choice is invalid or a rejecting configuration is encountered, go to step 4. If an accepting configuration is encountered, accept.
- ④ Replace the string on tape 3 with the next string (first short strings then longer ones, strings of same length in lexicographic order) and go to step 2.

□

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## Nondeterministic TMs no More Powerful than DTMs

Wouldn't it be easier to do a depth-first search for an accepting configuration in the computation tree?  
Why don't we do this and e.g. first entirely explore the first branch of the tree?



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## NTMs and DTMs are Equally Powerful

### Theorem

*A language is Turing-recognizable iff some nondeterministic Turing machine recognizes it.*

### Proof.

“ $\Rightarrow$ ”: Any DTM can be cast as a NTM.

“ $\Leftarrow$ ”: Previous theorem

□

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## B12.4 Summary

## Summary

We have seen several variants of Turing machines:

- ▶ Deterministic TM with head movements left or right
- ▶ Deterministic TM with head movements left, right or **neutral**
- ▶ **Multitape** Turing machines
- ▶ **Nondeterministic** Turing machines

**All variants recognize the same languages.**