Theory of Computer Science

B11. Turing Machines I

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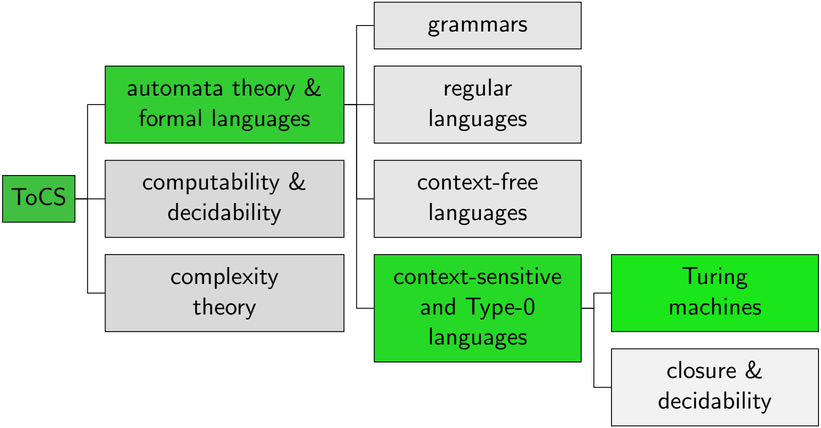
B11.1 Turing Machines

Content of the Course



Automata for Type-1 and Type-0 Languages?

Finite automata
recognize exactly the regular languages,
push-down automata exactly the
context-free languages. Are there
automata models for context-sensitive
and type-0 languages?

Yes! $\rightsquigarrow$ Turing machines

German: Turingmaschinen

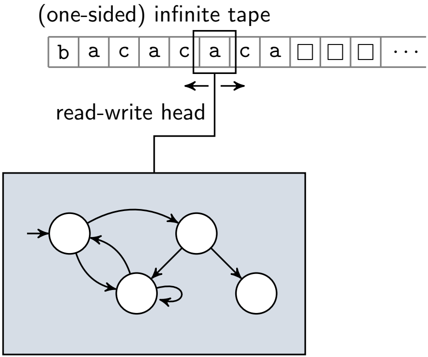
Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

Alan Turing (1912–1954)

Picture courtesy of Jon Callas / wikimedia commons

- British logician, mathematician, cryptanalyst and computer scientist
- most important work (for us): On Computable Numbers, with an Application to the Entscheidungsproblem
  $\rightsquigarrow$ Turing machines
- collaboration on Enigma decryption
- conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- Turing award most important science award in computer science

Turing Machines: Conceptually



Turing Machine: Definition

Definition (Deterministic Turing Machine)

A (deterministic) Turing machine (DTM) is given by a 7-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{accept}}, q_{\mathrm{reject}} \rangle$, where Q, Σ, Γ are all finite sets and

- Q is the set of states,
- Σ is the input alphabet, not containing the blank symbol $\square$,
- Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- $\delta : (Q \setminus \{q_{\mathrm{accept}}, q_{\mathrm{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
- $q_0 \in Q$ is the start state,
- $q_{\mathrm{accept}} \in Q$ is the accept state,
- $q_{\mathrm{reject}} \in Q$ is the reject state, where $q_{\mathrm{accept}} \neq q_{\mathrm{reject}}$.

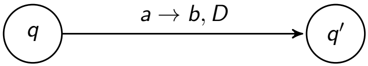
Turing Machine: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ be a DTM.

What is the Intuitive Meaning of the Transition Function δ?

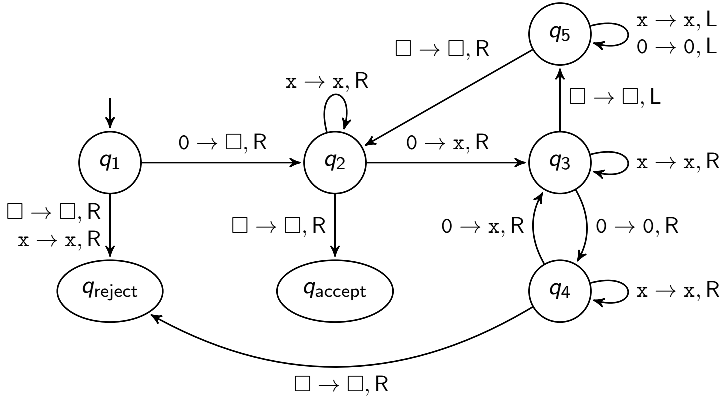
$\delta(q, a) = \langle q', b, D \rangle$:

- If M is in state q and reads a, then
- M transitions to state q' in the next step,
- replacing a with b,
- and moving the head in direction $D \in \{\mathrm{L}, \mathrm{R}\}$, where:
  - R: one step to the right,
  - L: one step to the left, except if the head is on the left-most cell of the tape in which case there is no movement



Deterministic Turing Machine: Example

$$\langle \{q_1, \ldots, q_5, q_{\text{accept}}, q_{\text{reject}}\}, \{0\}, \{0, \mathtt{x}, \square\}, \delta, q_1, q_{\text{accept}}, q_{\text{reject}} \rangle$$



Turing Machine: Configuration

Definition (Configuration of a Turing Machine)

A configuration of a Turing machine
$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$
is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

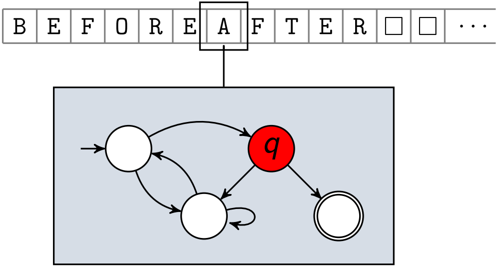
- the non-empty or already visited part of the tape contains the word $w_1 w_2$,
- the read-write head is on the first symbol of w_2, and
- the TM is in state q.

German: Konfiguration

Turing Machine Configurations: Example

Example

configuration $\langle \texttt{BEFORE}, q, \texttt{AFTER}\square\square \rangle$.



Turing Machine Configurations: Start Configuration

Initially

- the TM is in start state q_0,
- the head is on the first tape cell, and
- the tape contains the input word w followed by an infinite number of $\square$ entries.

The corresponding start configuration is $\langle \varepsilon, q_0, w \rangle$ if $w \neq \varepsilon$ and $\langle \varepsilon, q_0, \square \rangle$ if $w = \varepsilon$.

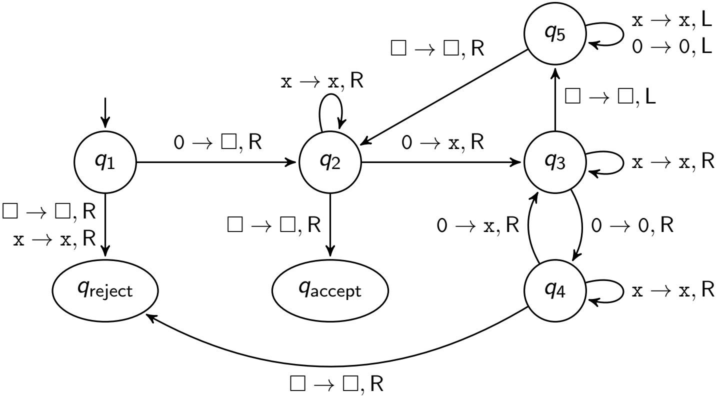
Turing Machine: Step

Definition (Transition/Step of a Turing Machine)

A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ transitions from configuration c to configuration c' in one step ($c \vdash_M c'$) according to the following rules:

- $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m c, q', b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{R} \rangle$, $m \geq 0$, $n \geq 2$
- $\langle a_1 \dots a_m, q, b_1 \rangle \vdash_M \langle a_1 \dots a_m c, q', \square \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{R} \rangle$, $m \geq 0$
- $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_{m-1}, q', a_m c b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{L} \rangle$, $m \geq 1$, $n \geq 1$
- $\langle \varepsilon, q, b_1 \dots b_n \rangle \vdash_M \langle \varepsilon, q', c b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{L} \rangle$, $n \geq 1$

Step: Exercise (Slido)



$\langle \square\mathtt{x}, q_3, 00 \rangle \vdash \,?$

DTM: Accepted Words

Intuitively, a DTM accepts a word if its computation terminates in the accept state.

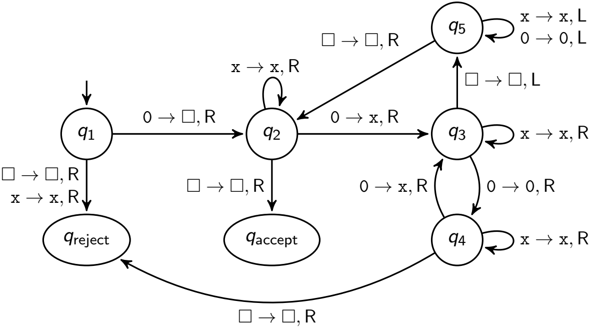
Definition (Words Accepted by a DTM)

DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ accepts the word $w = a_1 \dots a_n$ if there is a sequence of configurations $c_0, \dots, c_k$ with

1. c_0 is the start configuration of M on input w,
2. $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, k-1\}$, and
3. c_k is an accepting configuration, i. e., a configuration with state q_{accept}.

Accepted Word: Example (Slido)

Does this Turing machine accept input 0000?



DTM: Recognized Language

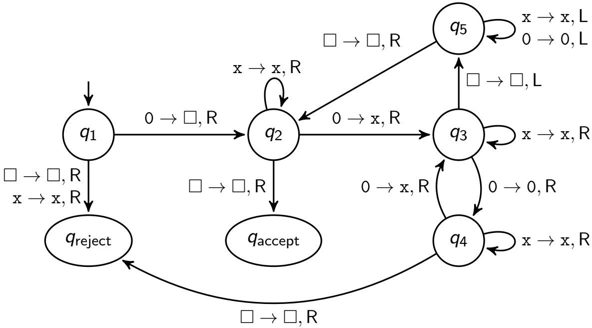
Definition (Language Recognized by a DTM)

Let M be a deterministic Turing Machine
The language recognized by M (or the language of M) is defined as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Definition (Turing-recognizable Language)

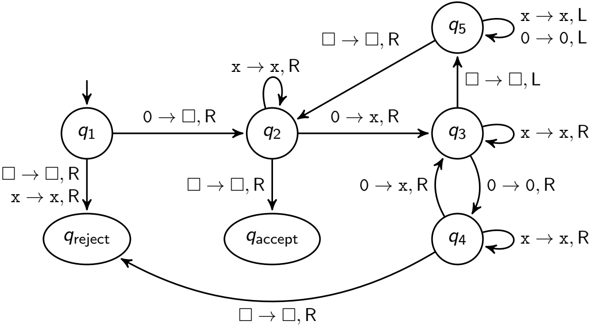
We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

Turing Machine: Example



1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, accept.
3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
4. Return the head to the left end of the tape and go to stage 1.

Recognized Language: Example



What language does the Turing machine recognize?

Deciders

- A Turing machine either fails to accept an input
  - because it rejects it (entering q_{reject}) or
  - because it loops ($=$ does not halt).
- A Turing machine that halts on all inputs (entering q_{reject} or q_{accept}) is called a decider.
- A decider that recognizes some language also is said to decide the language.

Definition (Turing-decidable Language)

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

Exercise (if time)

Specify the state diagram of a DTM that decides language

$$L = \{w\#w \mid w \in \{0,1\}^*\}.$$

Feel free to solve this together with your neighbour.

B11.2 Summary

Summary

- Turing machines only have finitely many states but an unbounded tape as "memory".
- Alan Turing proposed them as a mathematical model for arbitrary algorithmic computations.
- In this role, we will revisit them in the parts on computability and complexity theory.