Theory of Computer Science

B5. Regular Languages: Closure Properties and Decidability

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B5.1 Introduction

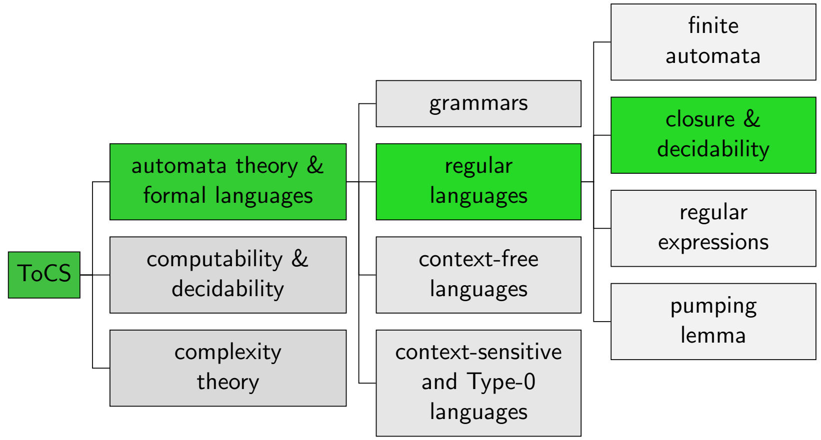
Further Analysis

We can convert freely between regular grammars, DFAs and NFAs.
So don't let's analyse them individually but instead focus on the corresponding class of regular languages:

- With what operations can we "combine" regular languages and the result is again a regular language?
  E.g. is the intersection of two regular languages regular?
- What general questions can we resolve algorithmically for any regular language?
  E.g. is there an algorithm that takes a regular grammars and a word as input and returns whether the word is in the generated language?

B5.2 Closure Properties

Content of the Course



Closure Properties

How can we combine regular languages so that the result is guaranteed to be regular as well?

Picture courtesy of stockimages / FreeDigitalPhotos.net

Concatenation of Languages

Concatenation

- For two languages L_1 (over Σ_1) and L_2 (over Σ_2), the concatenation of L_1 and L_2 is the language $L_1L_2 = \{w_1w_2 \in (\Sigma_1 \cup \Sigma_2)^* \mid w_1 \in L_1, w_2 \in L_2\}$.
- $L_1 = \{\text{Pancake}, \text{Waffle}\}$
  $L_2 = \{\text{withIceCream}, \text{withMushrooms}, \text{withCheese}\}$
  $L_1L_2 = \{\text{PancakewithIceCream}, \text{PancakewithMushrooms}, \text{PancakewithCheese}, \text{WafflewithIceCream}, \text{WafflewithMushrooms}, \text{WafflewithCheese}\}$

German: Produkt

Kleene Star

Kleene star

- For language L define
  - $L^0 = \{\varepsilon\}$
  - $L^1 = L$
  - $L^{i+1} = L^i L$ for $i \in \mathbb{N}_{>0}$
- Definition of (Kleene) star on L: $L^* = \bigcup_{i \geq 0} L^i$.
- $L = \{\text{ding}, \text{dong}\}$
  $L^* = \{\varepsilon, \text{ding}, \text{dong}, \text{dingding}, \text{dingdong}, \text{dongding}, \text{dongdong}, \text{dingdingding}, \text{dingdingdong}, \dots\}$

German: (Kleen)-Stern

Set Operations

Let L and L' be regular languages over Σ and Σ', respectively.

Languages are just sets of words, so we can also consider the standard set operations:

- union $L \cup L' = \{w \mid w \in L \text{ or } w \in L'\}$ over $\Sigma \cup \Sigma'$
- intersection $L \cap L' = \{w \mid w \in L \text{ and } w \in L'\}$ over $\Sigma \cap \Sigma'$
- complement $\bar{L} = \{w \in \Sigma^* \mid w \notin L\}$ over Σ

Closure Properties

General terminology: What do we mean with closure?

Definition (Closure)

Let $\mathcal{K}$ be a class of languages.

Then $\mathcal{K}$ is closed...

- ... under union if $L, L' \in \mathcal{K}$ implies $L \cup L' \in \mathcal{K}$
- ... under intersection if $L, L' \in \mathcal{K}$ implies $L \cap L' \in \mathcal{K}$
- ... under complement if $L \in \mathcal{K}$ implies $\bar{L} \in \mathcal{K}$
- ... under concatenation if $L, L' \in \mathcal{K}$ implies $LL' \in \mathcal{K}$
- ... under star if $L \in \mathcal{K}$ implies $L^* \in \mathcal{K}$

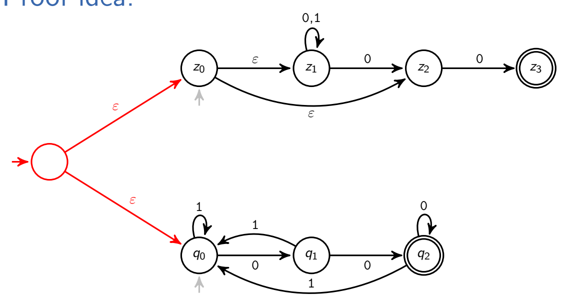
German: Abgeschlossenheit, $\mathcal{K}$ ist abgeschlossen unter Vereinigung, Schnitt, Komplement, Produkt, Stern

Closure Properties of Regular Languages: Union

Theorem

The regular languages are closed under union.

Proof idea:



Closure Properties of Regular Languages: Union

Proof.

Let L_1, L_2 be regular languages.

Let $M_1 = \langle Q_1, \Sigma_1, \delta_1, q_1, F_1 \rangle$ and $M_2 = \langle Q_2, \Sigma_2, \delta_2, q_2, F_2 \rangle$ be NFAs with $\mathcal{L}(M_1) = L_1$ and $\mathcal{L}(M_2) = L_2$. W.l.o.g. $Q_1 \cap Q_2 = \emptyset$.

Then NFA $M = \langle Q, \Sigma_1 \cup \Sigma_2, \delta, q_0, F_1 \cup F_2 \rangle$ with

- $q_0 \notin Q_1 \cup Q_2$ and
- $Q = \{q_0\} \cup Q_1 \cup Q_2$,
- for all $q \in Q, a \in \Sigma_1 \cup \Sigma_2 \cup \{\varepsilon\}$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma_1 \cup \{\varepsilon\} \\ \delta_2(q, a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma_2 \cup \{\varepsilon\} \\ \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \emptyset & \text{otherwise} \end{cases}$$

recognizes $L_1 \cup L_2$. □

Closure Properties of Regular Languages: Concatenation

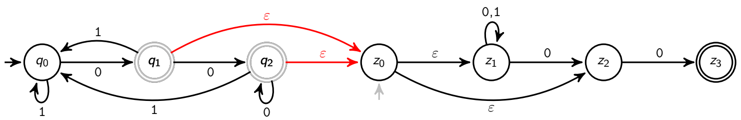
The proof idea for the closure under concatenation is very similar to the one for union. Can you figure it out yourself?

Closure Properties of Regular Languages: Concatenation

Theorem

The regular languages are closed under concatenation.

Proof idea:



Closure Properties of Regular Languages: Concatenation

Proof.

Let L_1, L_2 be regular languages.

Let $M_1 = \langle Q_1, \Sigma_1, \delta_1, q_1, F_1 \rangle$ and $M_2 = \langle Q_2, \Sigma_2, \delta_2, q_2, F_2 \rangle$ be NFAs with $\mathcal{L}(M_1) = L_1$ and $\mathcal{L}(M_2) = L_2$. W.l.o.g. $Q_1 \cap Q_2 = \emptyset$.

Then NFA $M = \langle Q_1 \cup Q_2, \Sigma_1 \cup \Sigma_2, \delta, q_1, F_2 \rangle$ with

- for all $q \in Q, a \in \Sigma_1 \cup \Sigma_2 \cup \{\varepsilon\}$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \setminus F_1 \text{ and } a \in \Sigma_1 \cup \{\varepsilon\} \\ \delta_1(q, a) & \text{if } q \in F_1 \text{ and } a \in \Sigma_1 \\ \delta_1(q, a) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q, a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma_2 \cup \{\varepsilon\} \\ \emptyset & \text{otherwise} \end{cases}$$

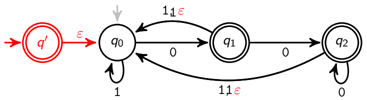
recognizes $L_1 L_2$. □

Closure Properties of Regular Languages: Star

Theorem

The regular languages are closed under star.

Proof idea:



Closure Properties of Regular Languages: Star

Proof.

Let L be a regular language.

Let $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ be an NFA with $\mathcal{L}(M) = L$.

Then NFA $M' = \langle Q', \Sigma, \delta', q'_0, F \cup \{q'\}\rangle$ with

- $q'_0 \notin Q$,
- $Q' = Q \cup \{q'_0\}$, and
- for all $q \in Q', a \in \Sigma \cup \{\varepsilon\}$

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } q \in Q \setminus F \\ \delta(q, a) & \text{if } q \in F \text{ and } a \in \Sigma \\ \delta(q, a) \cup \{q_0\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_0\} & \text{if } q = q'_0 \text{ and } a = \varepsilon \\ \emptyset & \text{otherwise} \end{cases}$$

recognizes L^*. □

Closure Properties of Regular Languages: Complement

Theorem

The regular languages are closed under complement.

Proof.

Let L be a regular language.

Let $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ be a DFA with $\mathcal{L}(M) = L$.

Then $M' = \langle Q, \Sigma, \delta, q_0, Q \setminus F\rangle$ is a DFA with $\mathcal{L}(M') = \bar{L}$. □

Closure Properties of Regular Languages: Intersection

Theorem

The regular languages are closed under intersection.

Proof.

Let L_1, L_2 be regular languages.

Let $M_1 = \langle Q_1, \Sigma_1, \delta_1, q_{01}, F_1\rangle$ and $M_2 = \langle Q_2, \Sigma_2, \delta_2, q_{02}, F_2\rangle$ be DFAs with $\mathcal{L}(M_1) = L_1$ and $\mathcal{L}(M_2) = L_2$.

The product automaton

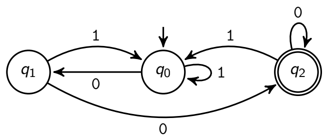
$$M = \langle Q_1 \times Q_2, \Sigma_1 \cap \Sigma_2, \delta, \langle q_{01}, q_{02}\rangle, F_1 \times F_2\rangle$$

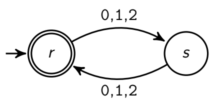
with $\delta(\langle q_1, q_2\rangle, a) = \langle \delta_1(q_1, a), \delta_2(q_2, a)\rangle$

accepts $\mathcal{L}(M) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$. □

German: Kreuzproduktautomat

Product Automaton: Example





Product Automaton: Blackboard

Closure Properties of Regular Languages

In summary...

Theorem

The regular languages are closed under:

- *union*
- *intersection*
- *complement*
- *concatenation*
- *star*

B5.3 Decidability

Decision Problems and Decidability (1)

"Intuitive Definition:" Decision Problem, Decidability

A decision problem is an algorithmic problem where

- for a given input
- an algorithm determines if the input has a given property
- and then produces the output "yes" or "no" accordingly.

A decision problem is decidable if an algorithm for it
(that always terminates and gives the correct answer) exists.

Note: "exists" $\neq$ "is known"

German: Entscheidungsproblem, Eingabe, Eigenschaft, Ausgabe, entscheidbar

Decision Problems and Decidability (2)

Notes:

- not a formal definition: we did not formally define "algorithm", "input", "output" etc. (which is not trivial)
- lack of a formal definition makes it difficult to prove that something is not decidable

$\rightsquigarrow$ studied thoroughly in the next part of the course

Decision Problems: Example

For now we describe decision problems in a semi-formal "given" / "question" way:

Example (Emptiness Problem for Regular Languages)

The emptiness problem $\mathrm{P}_\emptyset$ for regular languages is the following problem:

Given: regular grammar G
Question: Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

Word Problem

Definition (Word Problem for Regular Languages)

The word problem $\mathrm{P}_\in$ for regular languages is:

Given: regular grammar G with alphabet Σ and word $w \in \Sigma^*$
Question: Is $w \in \mathcal{L}(G)$?

German: Wortproblem (für reguläre Sprachen)

Decidability: Word Problem

Theorem

The word problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.
(The proofs in Chapter B4 describe a possible method.)

Simulate M on input w. The simulation ends after $|w|$ steps.

The DFA M is in an accept state after this iff $w \in \mathcal{L}(G)$.
Return "yes" or "no" accordingly. $\square$

Emptiness Problem

Definition (Emptiness Problem for Regular Languages)

The emptiness problem $\mathrm{P}_{\emptyset}$ for regular languages is:

Given: regular grammar G
Question: Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

Decidability: Emptiness Problem

Theorem

The emptiness problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $\mathcal{L}(G) = \emptyset$ iff in the transition diagram of M there is no path from the start state to any accept state.

This can be checked with standard graph algorithms (e.g., breadth-first search). □

Finiteness Problem

Definition (Finiteness Problem for Regular Languages)

The finiteness problem P_{∞} for regular languages is:

Given: regular grammar G
Question: Is $|\mathcal{L}(G)| < \infty$?

German: Endlichkeitsproblem

Decidability: Finiteness Problem

Theorem

The finiteness problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $|\mathcal{L}(G)| = \infty$ iff in the transition diagram of M there is a cycle that is reachable from the start state and from which an accept state can be reached.

This can be checked with standard graph algorithms. □

Intersection Problem

Definition (Intersection Problem for Regular Languages)

The intersection problem $\mathrm{P}_\cap$ for regular languages is:

Given: regular grammars G and G'

Question: Is $\mathcal{L}(G) \cap \mathcal{L}(G') = \emptyset$?

German: Schnittproblem

Decidability: Intersection Problem

Theorem

The intersection problem for regular languages is decidable.

Proof.

Using the closure of regular languages under intersection, we can construct (e.g., by converting to DFAs, constructing the product automaton, then converting back to a grammar) a grammar G'' with $\mathcal{L}(G'') = \mathcal{L}(G) \cap \mathcal{L}(G')$ and use the algorithm for the emptiness problem $\mathrm{P}_\emptyset$. □

Equivalence Problem

Definition (Equivalence Problem for Regular Languages)

The equivalence problem $\mathrm{P}_=$ for regular languages is:

Given: regular grammars G and G'

Question: Is $\mathcal{L}(G) = \mathcal{L}(G')$?

German: Äquivalenzproblem

Decidability: Equivalence Problem

Theorem

The equivalence problem for regular languages is decidable.

Proof.

In general for languages L and L', we have

$$L = L' \text{ iff } (L \cap \bar{L}') \cup (\bar{L} \cap L') = \emptyset.$$

The regular languages are closed under intersection, union and complement, and we know algorithms for these operations.

We can therefore construct a grammar for $(L \cap \bar{L}') \cup (\bar{L} \cap L')$ and use the algorithm for the emptiness problem $\mathrm{P}_\emptyset$. □

B5.4 Summary

Summary

- The regular languages are closed under all usual operations (union, intersection, complement, concatenation, star).
- All usual decision problems (word problem, emptiness, finiteness, intersection, equivalence) are decidable for regular languages.