Theory of Computer Science

B4. Finite Automata: Characterization

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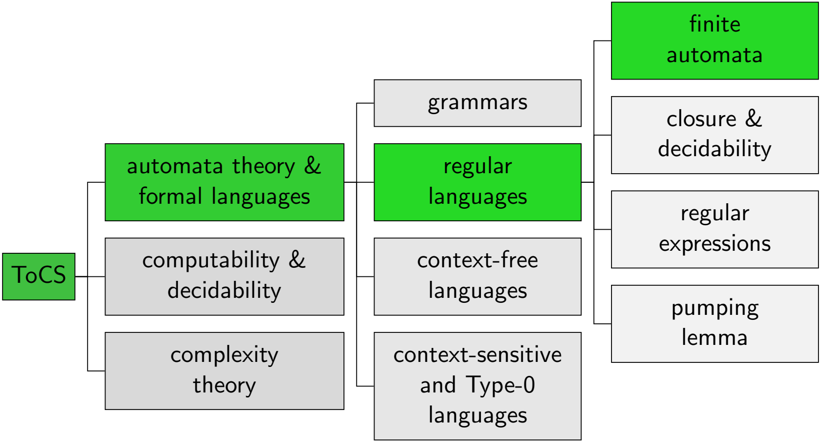
Content of the Course



Introduction

Finite Automata

Last chapter:

- Two kinds of finite automata: DFAs and NFAs.
- DFAs can be seen as a special case of NFAs.

Finite Automata

Last chapter:

- Two kinds of finite automata: DFAs and NFAs.
- DFAs can be seen as a special case of NFAs.

Questions for today:

- Are there languages that can only be recognized by one kind of finite automaton (but not the other)?
- Can we characterize the languages that DFAs/NFAs can recognize, e.g. within the Chomsky hierarchy?

DFAs vs. NFAs

DFAs are No More Powerful than NFAs

Observation

Every language recognized by a DFA is also recognized by an NFA.

We can transform a DFA into an NFA by replacing every transition $\delta(q, a) = q'$ with $\delta(q, a) = \{q'\}$.

Question

DFAs are
no more powerful than NFAs.
But are there languages
that can be recognized
by an NFA but not by a DFA?

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NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

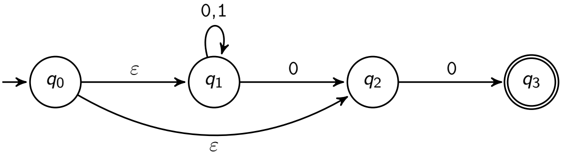
NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

The proof of the theorem is constructive and shows how we can convert an NFA to an equivalent DFA. Let's first have a look at the idea by means of an example (on the blackboard).

Conversion of an NFA to an Equivalent DFA: Example



NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

Proof.

For every NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ we can construct a DFA $M' = \langle Q', \Sigma, \delta', q'_0, F' \rangle$ with $\mathcal{L}(M) = \mathcal{L}(M')$.
Here M' is defined as follows:

- $Q' := \mathcal{P}(Q)$ (the power set of Q)
- $q'_0 := E(q_0)$
- $F' := \{\mathcal{Q} \subseteq Q \mid \mathcal{Q} \cap F \neq \emptyset\}$
- For all $\mathcal{Q} \in Q'$: $\delta'(\mathcal{Q}, a) := \bigcup_{q \in \mathcal{Q}} \bigcup_{q' \in \delta(q,a)} E(q')$

...

NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

Proof (continued).

For every $w = a_1 a_2 \dots a_n \in \Sigma^*$:

$w \in \mathcal{L}(M)$
iff there is a sequence of states $p_0, p_1, \dots, p_n$ with
  $p_0 \in E(q_0)$, $p_n \in F$ and
  $p_i \in \bigcup_{q \in \delta(p_{i-1}, a_i)} E(q)$ for all $i \in \{1, \dots, n\}$
iff there is a sequence of subsets $\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_n$ with
  $\mathcal{Q}_0 = q'_0$, $\mathcal{Q}_n \in F'$ and $\delta'(\mathcal{Q}_{i-1}, a_i) = \mathcal{Q}_i$ for all $i \in \{1, \dots, n\}$
iff $w \in \mathcal{L}(M')$ □

NFAs are More Compact than DFAs

Example

For $k \geq 1$ consider the language

$L_k = \{w \in \{0,1\}^* \mid |w| \geq k \text{ and the } k\text{-th last symbol of } w \text{ is } 0\}$.

NFAs are More Compact than DFAs

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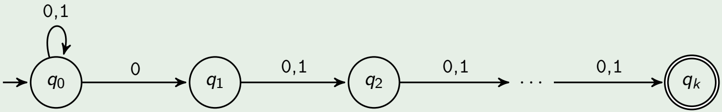
The language L_k can be recognized by an NFA with $k+1$ states:



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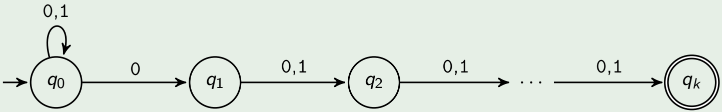

There is no DFA with less than 2^k states that recognizes L_k (without proof).

NFAs are More Compact than DFAs

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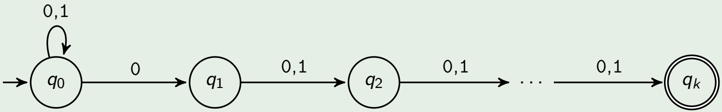

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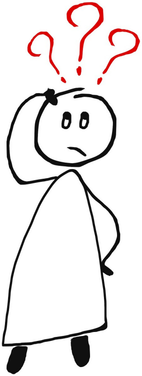
NFAs can often represent languages more compactly than DFAs.

Questions



Questions?

Finite Automata vs. Regular Languages

Languages Recognized by DFAs are Regular

Theorem

Every language recognized by a DFA is regular (type 3).

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Proof.

Let $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA.
We define a regular grammar G with $\mathcal{L}(G) = \mathcal{L}(M)$.

Define $G = \langle Q, \Sigma, R, q_0 \rangle$ where R contains

- a rule $q \to aq'$ for every $\delta(q, a) = q'$, and
- a rule $q \to \varepsilon$ for every $q \in F$.

(We can eliminate forbidden epsilon rules as described in Ch. B2.)

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Languages Recognized by DFAs are Regular

Theorem

Every language recognized by a DFA is regular (type 3).

Proof (continued).

For every $w = a_1 a_2 \dots a_n \in \Sigma^*$:

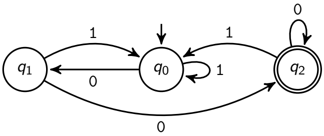
$w \in \mathcal{L}(M)$

iff there is a sequence of states $q'_0, q'_1, \dots, q'_n$ with
$q'_0 = q_0$, $q'_n \in F$ and $\delta(q'_{i-1}, a_i) = q'_i$ for all $i \in \{1, \dots, n\}$

iff there is a sequence of variables $q'_0, q'_1, \dots, q'_n$ with
q'_0 is start variable and we have $q'_0 \Rightarrow a_1 q'_1 \Rightarrow a_1 a_2 q'_2 \Rightarrow \dots \Rightarrow a_1 a_2 \dots a_n q'_n \Rightarrow a_1 a_2 \dots a_n$.

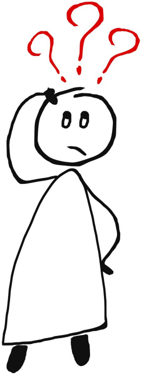
iff $w \in \mathcal{L}(G)$ □

Exercise



Specify a regular grammar that generates the language recognized by this DFA.

Questions



Questions?

Question

Is the inverse true as well:
for every regular language, is there a DFA that recognizes it? That is, are the languages recognized by DFAs exactly the regular languages?

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

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Is the inverse true as well:
for every regular language, is there a DFA that recognizes it? That is, are the languages recognized by DFAs exactly the regular languages?

Yes!
We will prove this via a detour.

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Regular Grammars are No More Powerful than NFAs

Theorem

For every regular grammar G there is an NFA M with $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof illustration:

Consider $G = \langle \{\mathsf{S}, \mathsf{A}, \mathsf{B}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S} \rangle$ with the following rules in R:

$$\mathsf{S} \to \varepsilon \qquad \mathsf{S} \to \mathtt{a}\mathsf{A} \qquad \mathsf{A} \to \mathtt{a}\mathsf{A} \qquad \mathsf{A} \to \mathtt{a}\mathsf{B}$$

$$\mathsf{A} \to \mathtt{a} \qquad \mathsf{B} \to \mathtt{b}\mathsf{B} \qquad \mathsf{B} \to \mathtt{b}$$

Regular Grammars are No More Powerful than NFAs

Theorem

For every regular grammar G there is an NFA M with $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof.

Let $G = \langle V, \Sigma, R, S\rangle$ be a regular grammar.
Define NFA $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ with

$$Q = V \cup \{X\}, \quad X \notin V$$

$$q_0 = S$$

$$F = \begin{cases} \{S, X\} & \text{if } S \to \varepsilon \in R \\ \{X\} & \text{if } S \to \varepsilon \notin R \end{cases}$$

$$B \in \delta(A, a) \text{ if } A \to aB \in R$$

$$X \in \delta(A, a) \text{ if } A \to a \in R$$

Regular Grammars are No More Powerful than NFAs

Theorem

For every regular grammar G there is an NFA M with $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof (continued).

For every $w = a_1 a_2 \dots a_n \in \Sigma^*$ with $n \geq 1$:

$w \in \mathcal{L}(G)$

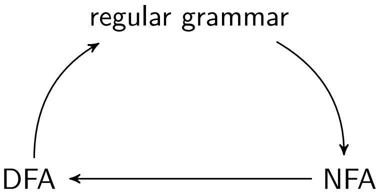
iff there is a sequence on variables $A_1, A_2, \dots, A_{n-1}$ with
$S \Rightarrow a_1 A_1 \Rightarrow a_1 a_2 A_2 \Rightarrow \dots \Rightarrow a_1 a_2 \dots a_{n-1} A_{n-1} \Rightarrow a_1 a_2 \dots a_n$.

iff there is a sequence of variables $A_1, A_2, \dots, A_{n-1}$ with
$A_1 \in \delta(S, a_1), A_2 \in \delta(A_1, a_2), \dots, X \in \delta(A_{n-1}, a_n)$.

iff $w \in \mathcal{L}(M)$.

Case $w = \varepsilon$ is also covered because $S \in F$ iff $S \to \varepsilon \in R$. □

Finite Automata and Regular Languages

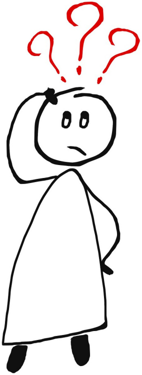


In particular, this implies:

Corollary

$\mathcal{L}$ regular $\Longleftrightarrow$ $\mathcal{L}$ is recognized by a DFA.
$\mathcal{L}$ regular $\Longleftrightarrow$ $\mathcal{L}$ is recognized by an NFA.

Questions



Questions?

Summary

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- DFAs and NFAs recognize the same languages.
- These are exactly the regular languages.