Theory of Computer Science

B3. Finite Automata

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University of Basel

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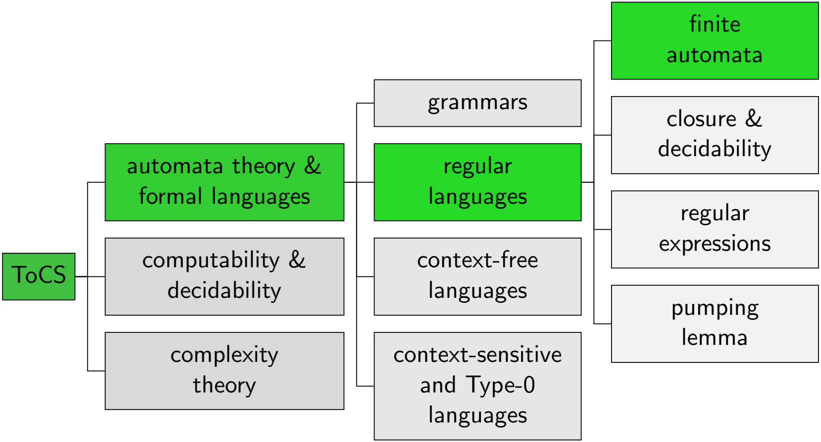
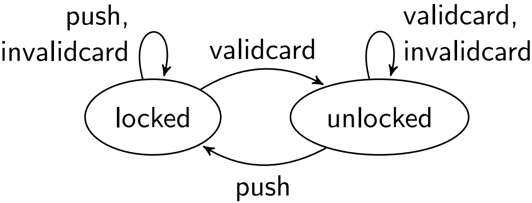
Introduction

Content of the Course



A Controller for a Turnstile

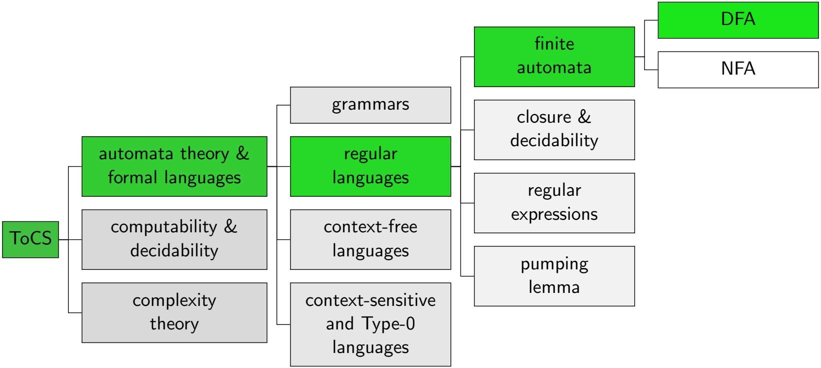
- simple access control
- card reader and push sensor
- card can either be valid or invalid



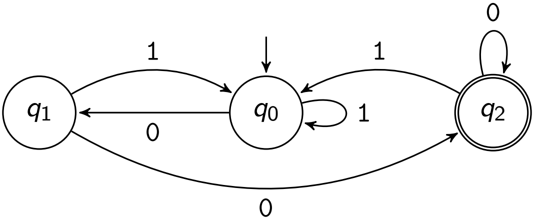
- Finite automata are a good model for computers with very limited memory.
  Where can the turnstile controller store information about what it has seen in the past?
- We will not consider automata that run forever but that process a finite input sequence and then classify it as accepted or not.

DFAs

Content of the Course



Finite Automaton: Example



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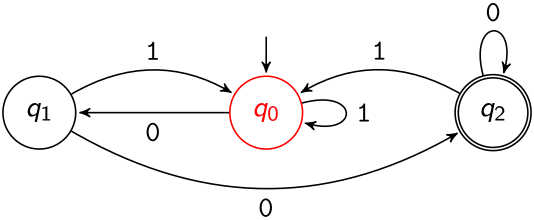

When reading the input 01100 the automaton visits the states q_0,

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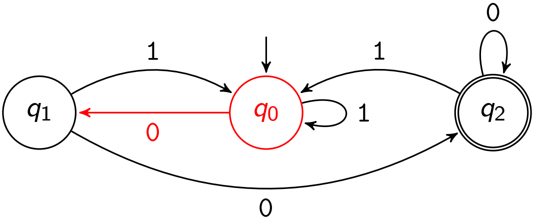

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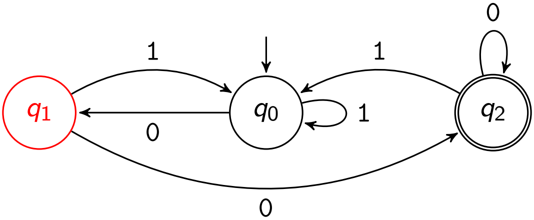

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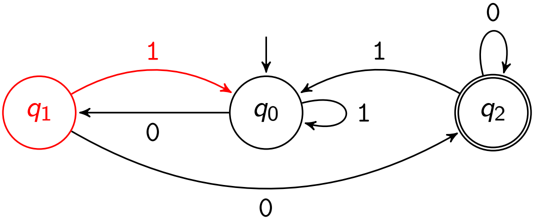

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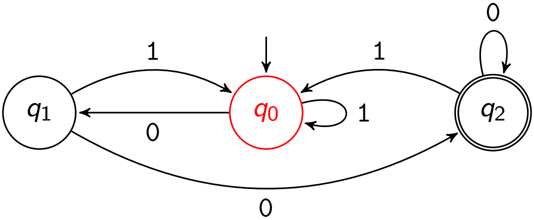

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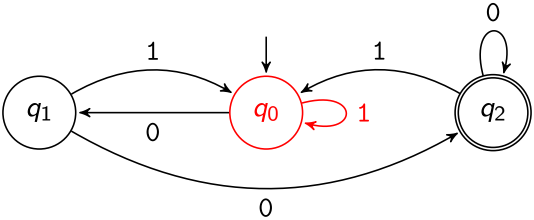

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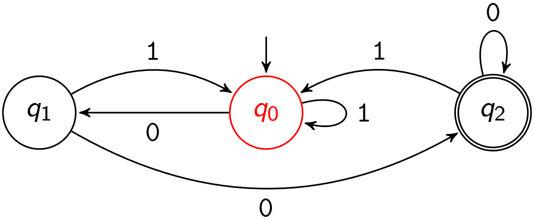

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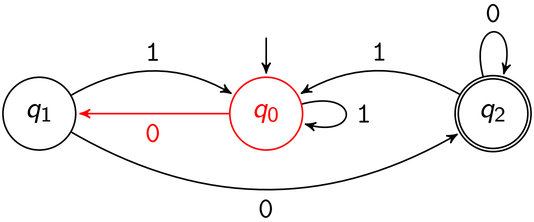

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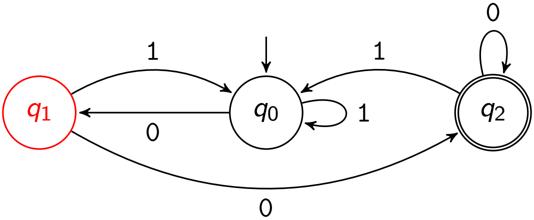

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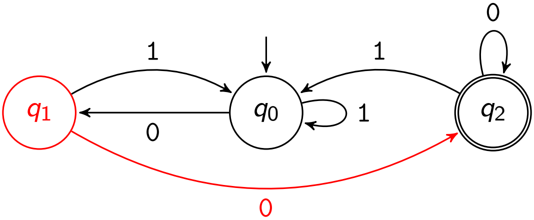

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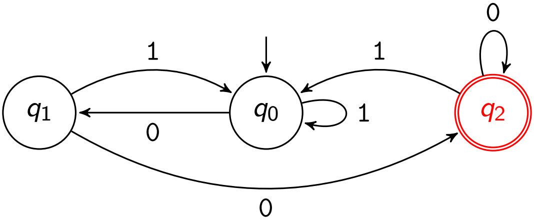
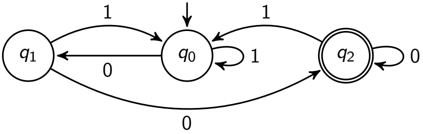


When reading the input 01100 the automaton visits the states q_0, q_1, q_0, q_0, q_1, q_2.

Finite Automata: Terminology and Notation



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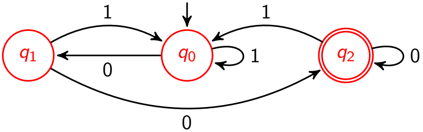
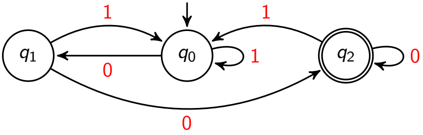


- states $Q = \{q_0, q_1, q_2\}$

Finite Automata: Terminology and Notation



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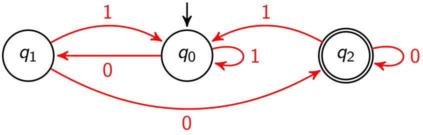
Finite Automata: Terminology and Notation



- states $Q = \{q_0, q_1, q_2\}$
- input alphabet $\Sigma = \{0, 1\}$
- transition function δ

$$\delta(q_0, 0) = q_1$$
$$\delta(q_0, 1) = q_0$$
$$\delta(q_1, 0) = q_2$$
$$\delta(q_1, 1) = q_0$$
$$\delta(q_2, 0) = q_2$$
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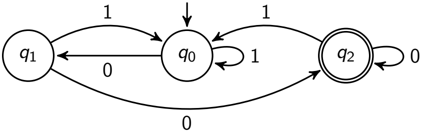
Finite Automata: Terminology and Notation



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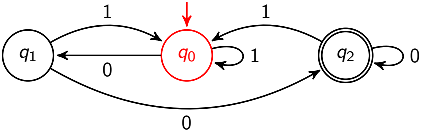
$$\delta(q_2, 1) = q_0$$

δ	0	1
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

table form of δ

Finite Automata: Terminology and Notation



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- start state q_0

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q_0	q_1	q_0
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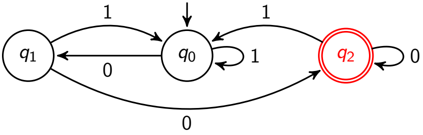
table form of δ

Finite Automata: Terminology and Notation



- states $Q = \{q_0, q_1, q_2\}$
- input alphabet $\Sigma = \{0, 1\}$
- transition function δ
- start state q_0
- accept states $\{q_2\}$

$$\delta(q_0, 0) = q_1$$
$$\delta(q_0, 1) = q_0$$
$$\delta(q_1, 0) = q_2$$
$$\delta(q_1, 1) = q_0$$
$$\delta(q_2, 0) = q_2$$
$$\delta(q_2, 1) = q_0$$

δ	0	1
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

table form of δ

Deterministic Finite Automaton: Definition

Definition (Deterministic Finite Automata)

A deterministic finite automaton (DFA) is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

- Q is the finite set of states
- Σ is the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states (or final states)

DFA: Accepted Words

Intuitively, a DFA accepts a word if its computation terminates in an accept state.

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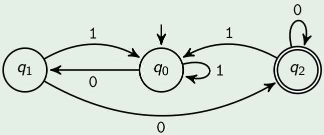
Definition (Words Accepted by a DFA)

DFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ accepts the word $w = a_1 \dots a_n$ if there is a sequence of states $q'_0, \dots, q'_n \in Q$ with

1. $q'_0 = q_0$,
2. $\delta(q'_{i-1}, a_i) = q'_i$ for all $i \in \{1, \dots, n\}$ and
3. $q'_n \in F$.

Example

Example

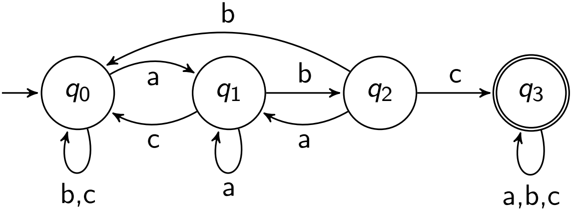


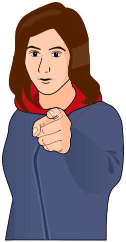
accepts:
00
10010100
01000

does not accept:
ε
1001010
010001

Exercise (slido)

Consider the following DFA:





Which of the following words does it accept?

- abc
- ababcb
- babbc

DFA: Recognized Language

Definition (Language Recognized by a DFA)

Let M be a deterministic finite automaton.
The language recognized by M is defined as
$\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

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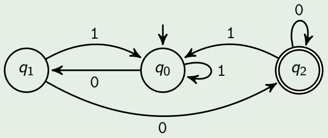

Example

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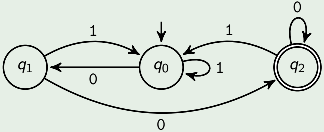
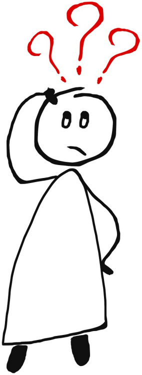


The DFA recognizes the language $\{w \in \{0,1\}^* \mid w \text{ ends with } 00\}$.

A Note on Terminology

- In the literature, "accept" and "recognize" are sometimes used synonymously or the other way around.
  DFA recognizes a word or accepts a language.
- We try to stay consistent using the previous definitions (following the text book by Sipser).

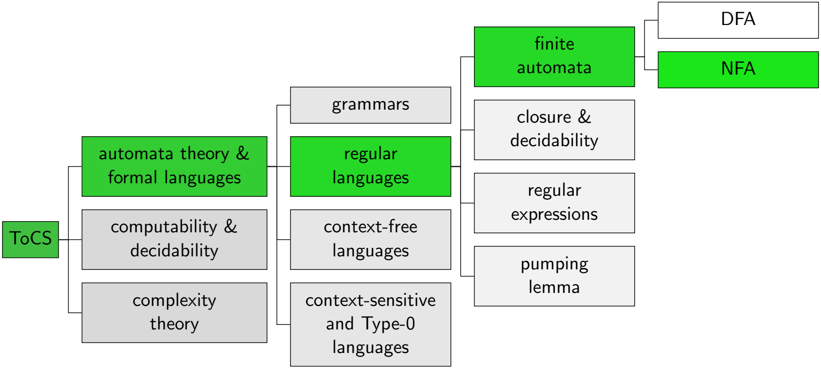
Questions



Questions?

NFAs

Content of the Course



Nondeterministic Finite Automata

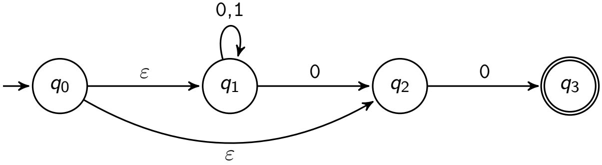
Why are DFAs called deterministic automata? What are nondeterministic automata, then?

Picture courtesy of stockimages / FreeDigitalPhotos.net

In what Sense is a DFA Deterministic?

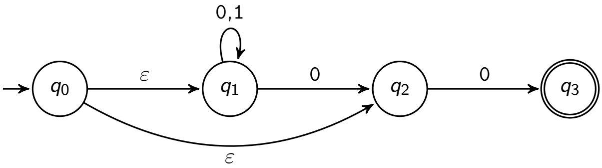
- A DFA has a single fixed state
  from which the computation starts.
- When a DFA is in a specific state and reads an input symbol,
  we know what the next state will be.
- For a given input, the entire computation is determined.
- This is a deterministic computation.

Nondeterministic Finite Automata: Example



differences to DFAs:

Nondeterministic Finite Automata: Example



differences to DFAs:

- transition function δ can lead to
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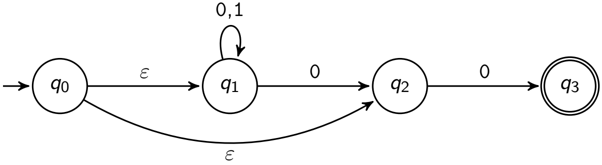

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- ε-transitions can be taken without "consuming" a symbol from the input

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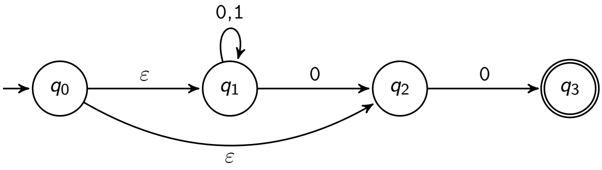

differences to DFAs:

- transition function δ can lead to zero or more successor states for the same $a \in \Sigma$
- ε-transitions can be taken without "consuming" a symbol from the input
- the automaton accepts a word if there is at least one accepting sequence of states

Nondeterministic Finite Automaton: Definition

Definition (Nondeterministic Finite Automata)

A nondeterministic finite automaton (NFA) is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

- Q is the finite set of states
- Σ is the input alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ is the transition function (mapping to the power set of Q)
- $q_0 \in Q$ is the start state
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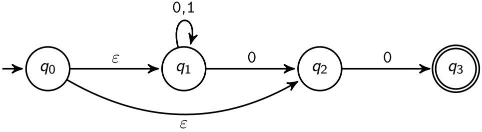
DFAs are (essentially) a special case of NFAs.

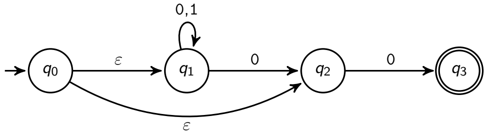
Accepting Computation: Example



$w = 0100$

⇝ computation tree on blackboard

Accepting Computation: Example



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ε-closure of a State

For a state $q \in Q$, we write $E(q)$ to denote the set of states that are reachable from q via ε-transitions in δ.

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For NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ and state $q \in Q$, state p is in the ε-closure $E(q)$ of q iff there is a sequence of states $q'_0, \dots, q'_n$ with

1. $q'_0 = q$,
2. $q'_i \in \delta(q'_{i-1}, \varepsilon)$ for all $i \in \{1, \dots, n\}$ and
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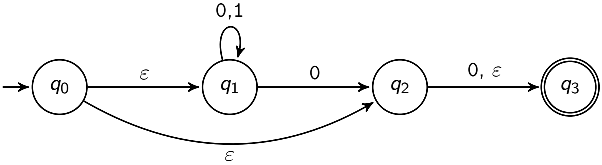
1. $q'_0 = q$,
2. $q'_i \in \delta(q'_{i-1}, \varepsilon)$ for all $i \in \{1, \ldots, n\}$ and
3. $q'_n = p$.

$q \in E(q)$ for every state q

Exercise (slido)

Consider the following NFA:



Which states are in the ε-closure $E(q_0)$?

- q_0
- q_1
- q_2
- q_3

NFA: Accepted Words

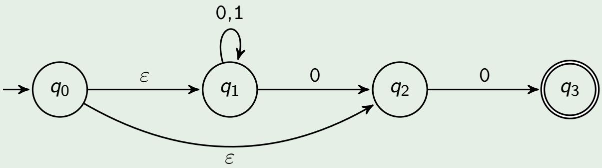
Definition (Words Accepted by an NFA)

NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ accepts the word $w = a_1 \dots a_n$ if there is a sequence of states $q'_0, \dots, q'_n \in Q$ with

1. $q'_0 \in E(q_0)$,
2. $q'_i \in \bigcup_{q \in \delta(q'_{i-1}, a_i)} E(q)$ for all $i \in \{1, \dots, n\}$ and
3. $q'_n \in F$.

Example: Accepted Words

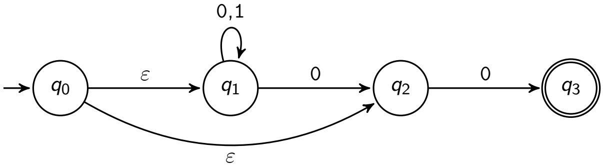
Example



accepts:
0
10010100
01000

does not accept:
ε
1001010
010001

Exercise (slido)



Does this NFA accept input 01010?

NFA: Recognized Language

Definition (Language Recognized by an NFA)

Let M be an NFA with input alphabet Σ.

The language recognized by M is defined as
$\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Example: Recognized Language

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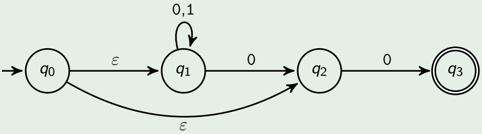

Example: Recognized Language

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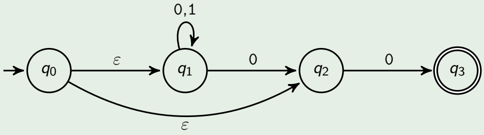
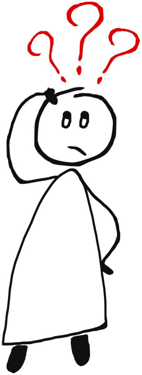


The NFA recognizes the language
$\{w \in \{0,1\}^* \mid w = 0 \text{ or } w \text{ ends with } 00\}$.

Questions



Questions?

Summary

Summary

- DFAs are automata where every state transition is uniquely determined.
- NFAs can have zero, one or more transitions for a given state and input symbol.
- NFAs can have ϵ-transitions that can be taken without reading a symbol from the input.
- NFAs accept a word if there is at least one accepting sequence of states.