

# Theory of Computer Science

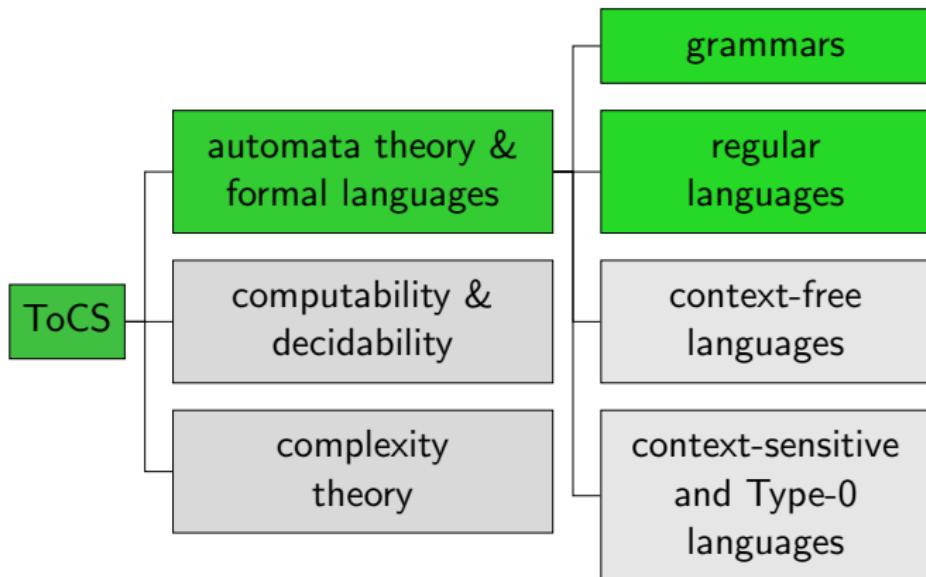
## B2. Regular Grammars: $\varepsilon$ -Rules

Gabriele Röger

University of Basel

March 3, 2025

# Content of the Course



# Recap

# Recap: Regular Grammars

## Definition (Regular Grammars)

A **regular grammar** is a 4-tuple  $\langle V, \Sigma, R, S \rangle$  with

- $V$  finite set of variables (nonterminal symbols)
- $\Sigma$  finite alphabet of terminal symbols with  $V \cap \Sigma = \emptyset$
- $R \subseteq (V \times (\Sigma \cup \Sigma V)) \cup \{\langle S, \varepsilon \rangle\}$  finite set of rules
- if  $S \rightarrow \varepsilon \in R$ , there is no  $X \in V, y \in \Sigma$  with  $X \rightarrow yS \in R$
- $S \in V$  start variable.

# Recap: Regular Grammars

## Definition (Regular Grammars)

A **regular grammar** is a 4-tuple  $\langle V, \Sigma, R, S \rangle$  with

- $V$  finite set of variables (nonterminal symbols)
- $\Sigma$  finite alphabet of terminal symbols with  $V \cap \Sigma = \emptyset$
- $R \subseteq (V \times (\Sigma \cup \Sigma V)) \cup \{\langle S, \varepsilon \rangle\}$  finite set of rules
- if  $S \rightarrow \varepsilon \in R$ , there is no  $X \in V, y \in \Sigma$  with  $X \rightarrow yS \in R$
- $S \in V$  start variable.

Rule  $X \rightarrow \varepsilon$  is only allowed if  $X = S$  and  
 $S$  never occurs in the right-hand side of a rule.

# Question (Slido)

With a regular grammar, how many steps does it take to derive a non-empty word (over  $\Sigma$ ) from the start variable?



# Recap: Regular Languages

A language is regular if it is generated by some regular grammar.

## Definition (Regular Language)

A language  $L \subseteq \Sigma^*$  is **regular**  
if there exists a regular grammar  $G$  with  $\mathcal{L}(G) = L$ .

# Epsilon Rules

# Regular Grammars

## Definition (Regular Grammars)

A **regular grammar** is a 4-tuple  $\langle V, \Sigma, R, S \rangle$  with

- $V$  finite set of variables (nonterminal symbols)
- $\Sigma$  finite alphabet of terminal symbols with  $V \cap \Sigma = \emptyset$
- $R \subseteq (V \times (\Sigma \cup \Sigma V)) \cup \{\langle S, \varepsilon \rangle\}$  finite set of rules
- if  $S \rightarrow \varepsilon \in R$ , there is no  $X \in V, y \in \Sigma$  with  $X \rightarrow yS \in R$
- $S \in V$  start variable.

Rule  $X \rightarrow \varepsilon$  is only allowed if  $X = S$  and  
 $S$  never occurs in the right-hand side of a rule.

**How restrictive is this?** If we don't restrict the usage of  $\varepsilon$  as  
right-hand side of a rule, what does this change?

# Our Plan

We are going to show that every grammar with rules

$$R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$$

generates a regular language.

# Question



This is much simpler!  
Why don't we define  
regular languages  
via such grammars?

# Question

Both variants (restricting the occurrence of  $\varepsilon$  on the right-hand side of rules or not) characterize exactly the regular languages.



In the following situations, which variant would you prefer?

- You want to prove something for all regular languages.
- You want to specify a grammar to establish that a certain language is regular.
- You want to write an algorithm that takes a grammar for a regular language as input.

# Our Plan

We are going to show that every grammar with rules

$$R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$$

generates a regular language.

- The proof will be **constructive**, i. e. it will tell us how to construct a regular grammar for a language that is given by such a more general grammar.
- Two steps:
  - ➊ Eliminate the start variable from the right-hand side of rules.
  - ➋ Eliminate forbidden occurrences of  $\varepsilon$ .

# Start Variable in Right-Hand Side of Rules

For every type-0 language  $L$  there is a grammar where the start variable does not occur on the right-hand side of any rule.

## Theorem

*For every grammar  $G = \langle V, \Sigma, R, S \rangle$  there is a grammar*

*$G' = \langle V', \Sigma, R', S \rangle$  with rules*

*$R' \subseteq (V' \cup \Sigma)^* V' (V' \cup \Sigma)^* \times (V' \setminus \{S\} \cup \Sigma)^*$  such that*

*$\mathcal{L}(G) = \mathcal{L}(G')$ .*

Note: this theorem is true for **all** grammars.

## Start Variable in Right-Hand Side of Rules: Example

Before we prove the theorem, let's illustrate its idea.

Consider  $G = \langle \{S, X\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$bS \rightarrow \varepsilon \quad S \rightarrow XabS \quad bX \rightarrow aSa \quad X \rightarrow abc$$

## Start Variable in Right-Hand Side of Rules: Example

Before we prove the theorem, let's illustrate its idea.

Consider  $G = \langle \{S, X\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$bS \rightarrow \varepsilon \quad S \rightarrow XabS \quad bX \rightarrow aSa \quad X \rightarrow abc$$

The new grammar has all original rules except that  $S$  is replaced with a new variable  $S'$  (allowing to derive everything from  $S'$  that could originally be derived from the start variable  $S$ ):

$$bS' \rightarrow \varepsilon \quad S' \rightarrow XabS' \quad bX \rightarrow aS'a \quad X \rightarrow abc$$

## Start Variable in Right-Hand Side of Rules: Example

Before we prove the theorem, let's illustrate its idea.

Consider  $G = \langle \{S, X\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$bS \rightarrow \varepsilon \quad S \rightarrow XabS \quad bX \rightarrow aSa \quad X \rightarrow abc$$

The new grammar has all original rules except that  $S$  is replaced with a new variable  $S'$  (allowing to derive everything from  $S'$  that could originally be derived from the start variable  $S$ ):

$$bS' \rightarrow \varepsilon \quad S' \rightarrow XabS' \quad bX \rightarrow aS'a \quad X \rightarrow abc$$

In addition, it has rules that allow to start from the original start variable but switch to  $S'$  after the first rule application:

$$S \rightarrow XabS'$$

# Start Variable in Right-Hand Side of Rules: Proof

## Proof.

Let  $G = \langle V, \Sigma, R, S \rangle$  be a grammar and  $S' \notin V$  be a new variable. Construct rule set  $R'$  from  $R$  as follows:

- for every rule  $r \in R$ , add a rule  $r'$  to  $R'$ , where  $r'$  is the result of replacing all occurrences of  $S$  in  $r$  with  $S'$ .
- for every rule  $S \rightarrow w \in R$ , add a rule  $S \rightarrow w'$  to  $R'$ , where  $w'$  is the result of replacing all occurrences of  $S$  in  $w$  with  $S'$ .

Then  $\mathcal{L}(G) = \mathcal{L}(\langle V \cup \{S'\}, \Sigma, R', S \rangle)$ .



# Start Variable in Right-Hand Side of Rules: Proof

## Proof.

Let  $G = \langle V, \Sigma, R, S \rangle$  be a grammar and  $S' \notin V$  be a new variable. Construct rule set  $R'$  from  $R$  as follows:

- for every rule  $r \in R$ , add a rule  $r'$  to  $R'$ , where  $r'$  is the result of replacing all occurrences of  $S$  in  $r$  with  $S'$ .
- for every rule  $S \rightarrow w \in R$ , add a rule  $S \rightarrow w'$  to  $R'$ , where  $w'$  is the result of replacing all occurrences of  $S$  in  $w$  with  $S'$ .

Then  $\mathcal{L}(G) = \mathcal{L}(\langle V \cup \{S'\}, \Sigma, R', S \rangle)$ . □

Note that the rules in  $R'$  are not fundamentally different from the rules in  $R$ . In particular:

- If  $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$  then  $R' \subseteq V' \times (\Sigma \cup \Sigma V' \cup \{\varepsilon\})$ .
- If  $R \subseteq V \times (V \cup \Sigma)^*$  then  $R' \subseteq V' \times (V' \cup \Sigma)^*$ .

# Epsilon Rules

## Theorem

*For every grammar  $G$  with rules  $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$  there is a regular grammar  $G'$  with  $\mathcal{L}(G) = \mathcal{L}(G')$ .*

## Epsilon Rules: Example

Let's again first illustrate the idea. Consider

$G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \quad S \rightarrow aX \quad X \rightarrow aX \quad X \rightarrow aY \quad Y \rightarrow bY \quad Y \rightarrow \varepsilon$$

## Epsilon Rules: Example

Let's again first illustrate the idea. Consider

$G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \quad S \rightarrow aX \quad X \rightarrow aX \quad X \rightarrow aY \quad Y \rightarrow bY \quad Y \rightarrow \varepsilon$$

- ① The start variable does not occur on a right-hand side. ✓

## Epsilon Rules: Example

Let's again first illustrate the idea. Consider

$G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \quad S \rightarrow aX \quad X \rightarrow aX \quad X \rightarrow aY \quad Y \rightarrow bY \quad Y \rightarrow \varepsilon$$

- ① The start variable does not occur on a right-hand side. ✓
- ② Determine the set of variables that can be replaced with the empty word:  $V_\varepsilon = \{S, Y\}$ .

## Epsilon Rules: Example

Let's again first illustrate the idea. Consider

$G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \quad S \rightarrow aX \quad X \rightarrow aX \quad X \rightarrow aY \quad Y \rightarrow bY \quad Y \rightarrow \varepsilon$$

- ① The start variable does not occur on a right-hand side. ✓
- ② Determine the set of variables that can be replaced with the empty word:  $V_\varepsilon = \{S, Y\}$ .
- ③ Eliminate forbidden rules: ~~X~~/~~H~~/~~E~~

## Epsilon Rules: Example

Let's again first illustrate the idea. Consider

$G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \quad S \rightarrow aX \quad X \rightarrow aX \quad X \rightarrow aY \quad Y \rightarrow bY \quad Y \rightarrow \varepsilon$$

- ① The start variable does not occur on a right-hand side. ✓
- ② Determine the set of variables that can be replaced with the empty word:  $V_\varepsilon = \{S, Y\}$ .
- ③ Eliminate forbidden rules: ~~X//H//F~~
- ④ If a variable from  $V_\varepsilon$  occurs in the right-hand side, add another rule that directly emulates a subsequent replacement with the empty word:  $X \rightarrow a$  and  $Y \rightarrow b$

# Epsilon Rules

## Theorem

*For every grammar  $G$  with rules  $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$  there is a regular grammar  $G'$  with  $\mathcal{L}(G) = \mathcal{L}(G')$ .*

## Proof.

Let  $G = \langle V, \Sigma, R, S \rangle$  be a grammar s.t.  $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$ .

Use the previous proof to construct grammar  $G' = \langle V', \Sigma, R', S \rangle$  s.t.  $R' \subseteq V' \times (\Sigma \cup \Sigma(V' \setminus \{S\}) \cup \{\varepsilon\})$  and  $\mathcal{L}(G') = \mathcal{L}(G)$ .

Let  $V_\varepsilon = \{A \mid A \rightarrow \varepsilon \in R'\}$ .

Let  $R''$  be the rule set that is created from  $R'$  by removing all rules of the form  $A \rightarrow \varepsilon$  ( $A \neq S$ ). Additionally, for every rule of the form  $B \rightarrow xA$  with  $A \in V_\varepsilon$ ,  $B \in V'$ ,  $x \in \Sigma$  we add a rule  $B \rightarrow x$  to  $R''$ .

Then  $G'' = \langle V', \Sigma, R'', S \rangle$  is regular and  $\mathcal{L}(G) = \mathcal{L}(G'')$ . □

# Questions



## Exercise (Slido)

Consider  $G = \langle \{S, X, Y\}, \{a, b\}, R, S \rangle$  with the following rules in  $R$ :

$$S \rightarrow \varepsilon \qquad S \rightarrow aX$$

$$X \rightarrow aX \qquad X \rightarrow aY$$

$$Y \rightarrow bY \qquad Y \rightarrow \varepsilon$$



- Is  $G$  a regular grammar?
- Is  $\mathcal{L}(G)$  regular?

# Summary

# Summary

- Regular grammars restrict the usage of  $\varepsilon$  in rules.
- This restriction is not necessary for the characterization of regular languages but convenient if we want to prove something for all regular languages.