Theory of Computer Science

B2. Regular Grammars: ε-Rules

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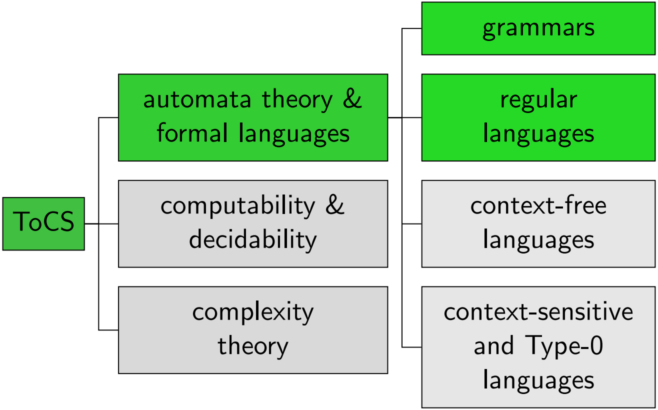
Content of the Course



Recap

Recap: Regular Grammars

Definition (Regular Grammars)

A regular grammar is a 4-tuple $\langle V, \Sigma, R, S\rangle$ with

- V finite set of variables (nonterminal symbols)
- Σ finite alphabet of terminal symbols with $V \cap \Sigma = \emptyset$
- $R \subseteq (V \times (\Sigma \cup \Sigma V)) \cup \{\langle S, \varepsilon\rangle\}$ finite set of rules
- if $S \to \varepsilon \in R$, there is no $X \in V, y \in \Sigma$ with $X \to yS \in R$
- $S \in V$ start variable.

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Rule $X \to \varepsilon$ is only allowed if $X = S$ and
S never occurs in the right-hand side of a rule.

Question (Slido)

With a regular grammar, how many steps does it take to derive a non-empty word (over Σ) from the start variable?

Recap: Regular Languages

A language is regular if it is generated by some regular grammar.

Definition (Regular Language)

A language $L \subseteq \Sigma^*$ is regular
if there exists a regular grammar G with $\mathcal{L}(G) = L$.

Epsilon Rules

Regular Grammars

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- $S \in V$ start variable.

Rule $X \rightarrow \varepsilon$ is only allowed if $X = S$ and
S never occurs in the right-hand side of a rule.

How restrictive is this? If we don't restrict the usage of ε as right-hand side of a rule, what does this change?

Our Plan

We are going to show that every grammar with rules

$R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$

generates a regular language.

Question

This is much simpler!
Why don't we define
regular languages
via such grammars?

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

Question

Both variants (restricting the occurrence of ε on the right-hand side of rules or not) characterize exactly the regular languages.

In the following situations, which variant would you prefer?

- You want to prove something for all regular languages.
- You want to specify a grammar to establish that a certain language is regular.
- You want to write an algorithm that takes a grammar for a regular language as input.

Our Plan

We are going to show that every grammar with rules

$R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$

generates a regular language.

- The proof will be constructive, i. e. it will tell us how to construct a regular grammar for a language that is given by such a more general grammar.
- Two steps:
  1. Eliminate the start variable from the right-hand side of rules.
  2. Eliminate forbidden occurrences of ε.

Start Variable in Right-Hand Side of Rules

For every type-0 language L there is a grammar where the start variable does not occur on the right-hand side of any rule.

Theorem

For every grammar $G = \langle V, \Sigma, R, S \rangle$ *there is a grammar* $G' = \langle V', \Sigma, R', S \rangle$ *with rules* $R' \subseteq (V' \cup \Sigma)^* V' (V' \cup \Sigma)^* \times (V' \setminus \{S\} \cup \Sigma)^*$ *such that* $\mathcal{L}(G) = \mathcal{L}(G')$.

Note: this theorem is true for all grammars.

Start Variable in Right-Hand Side of Rules: Example

Before we prove the theorem, let's illustrate its idea.
Consider $G = \langle \{S, X\}, \{a, b\}, R, S \rangle$ with the following rules in R:

$bS \rightarrow \varepsilon$ $\quad S \rightarrow XabS$ $\quad bX \rightarrow aSa$ $\quad X \rightarrow abc$

Start Variable in Right-Hand Side of Rules: Example

Before we prove the theorem, let's illustrate its idea.
Consider $G = \langle\{\mathsf{S}, \mathsf{X}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S}\rangle$ with the following rules in R:

$\mathtt{b}\mathsf{S} \to \varepsilon$ $\quad$ $\mathsf{S} \to \mathsf{X}\mathtt{ab}\mathsf{S}$ $\quad$ $\mathtt{b}\mathsf{X} \to \mathtt{a}\mathsf{S}\mathtt{a}$ $\quad$ $\mathsf{X} \to \mathtt{abc}$

The new grammar has all original rules except that S is replaced with a new variable S' (allowing to derive everything from S' that could originally be derived from the start variable S):

$\mathtt{b}\mathsf{S'} \to \varepsilon$ $\quad$ $\mathsf{S'} \to \mathsf{X}\mathtt{ab}\mathsf{S'}$ $\quad$ $\mathtt{b}\mathsf{X} \to \mathtt{a}\mathsf{S'}\mathtt{a}$ $\quad$ $\mathsf{X} \to \mathtt{abc}$

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In addition, it has rules that allow to start from the original start variable but switch to S' after the first rule application:

$\mathsf{S} \to \mathsf{X}\mathtt{ab}\mathsf{S'}$

Start Variable in Right-Hand Side of Rules: Proof

Proof.

Let $G = \langle V, \Sigma, R, S \rangle$ be a grammar and $S' \notin V$ be a new variable. Construct rule set R' from R as follows:

- for every rule $r \in R$, add a rule r' to R', where r' is the result of replacing all occurences of S in r with S'.
- for every rule $S \to w \in R$, add a rule $S \to w'$ to R', where w' is the result of replacing all occurences of S in w with S'.

Then $\mathcal{L}(G) = \mathcal{L}(\langle V \cup \{S'\}, \Sigma, R', S \rangle)$. □

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Then $\mathcal{L}(G) = \mathcal{L}(\langle V \cup \{S'\}, \Sigma, R', S \rangle)$. □

Note that the rules in R' are not fundamentally different from the rules in R. In particular:

- If $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$ then $R' \subseteq V' \times (\Sigma \cup \Sigma V' \cup \{\varepsilon\})$.
- If $R \subseteq V \times (V \cup \Sigma)^*$ then $R' \subseteq V' \times (V' \cup \Sigma)^*$.

Epsilon Rules

Theorem

For every grammar G with rules $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$ there is a regular grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Epsilon Rules: Example

Let's again first illustrate the idea.Consider $G = \langle\{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S}\rangle$ with the following rules in R:

$\mathsf{S} \to \varepsilon \qquad \mathsf{S} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{Y} \qquad \mathsf{Y} \to \mathtt{b}\mathsf{Y} \qquad \mathsf{Y} \to \varepsilon$

Epsilon Rules: Example

Let's again first illustrate the idea.Consider
$G = \langle\{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S}\rangle$ with the following rules in R:

$\mathsf{S} \to \varepsilon \qquad \mathsf{S} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{Y} \qquad \mathsf{Y} \to \mathtt{b}\mathsf{Y} \qquad \mathsf{Y} \to \varepsilon$

1. The start variable does not occur on a right-hand side. ✓

Epsilon Rules: Example

Let's again first illustrate the idea. Consider
$G = \langle\{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S}\rangle$ with the following rules in R:

$\mathsf{S} \to \varepsilon$ $\quad$ $\mathsf{S} \to \mathtt{a}\mathsf{X}$ $\quad$ $\mathsf{X} \to \mathtt{a}\mathsf{X}$ $\quad$ $\mathsf{X} \to \mathtt{a}\mathsf{Y}$ $\quad$ $\mathsf{Y} \to \mathtt{b}\mathsf{Y}$ $\quad$ $\mathsf{Y} \to \varepsilon$

1. The start variable does not occur on a right-hand side. ✓
2. Determine the set of variables that can be replaced with the empty word: $V_\varepsilon = \{S, Y\}$.

Epsilon Rules: Example

Let's again first illustrate the idea.Consider
$G = \langle \{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathsf{S} \rangle$ with the following rules in R:

$$\mathsf{S} \to \varepsilon \qquad \mathsf{S} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{X} \qquad \mathsf{X} \to \mathtt{a}\mathsf{Y} \qquad \mathsf{Y} \to \mathtt{b}\mathsf{Y} \qquad \mathsf{Y} \to \varepsilon$$

1. The start variable does not occur on a right-hand side. ✓
2. Determine the set of variables that can be replaced with the empty word: $V_\varepsilon = \{S, Y\}$.
3. Eliminate forbidden rules: ~~$\mathsf{Y} \to \varepsilon$~~

Epsilon Rules: Example

Let's again first illustrate the idea.Consider
$G = \langle\{S, X, Y\}, \{a, b\}, R, S\rangle$ with the following rules in R:

$$S \to \varepsilon \qquad S \to aX \qquad X \to aX \qquad X \to aY \qquad Y \to bY \qquad Y \to \varepsilon$$

1. The start variable does not occur on a right-hand side. ✓
2. Determine the set of variables that can be replaced with the empty word: $V_\varepsilon = \{S, Y\}$.
3. Eliminate forbidden rules: ~~$Y \to \varepsilon$~~
4. If a variable from V_ε occurs in the right-hand side, add another rule that directly emulates a subsequent replacement with the empty word: $X \to a$ and $Y \to b$

Epsilon Rules

Theorem

For every grammar G with rules $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$ there is a regular grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

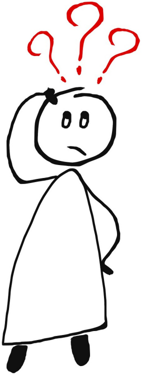
Proof.

Let $G = \langle V, \Sigma, R, S\rangle$ be a grammar s.t. $R \subseteq V \times (\Sigma \cup \Sigma V \cup \{\varepsilon\})$.
Use the previous proof to construct grammar $G' = \langle V', \Sigma, R', S\rangle$ s.t. $R' \subseteq V' \times (\Sigma \cup \Sigma(V' \setminus \{S\}) \cup \{\varepsilon\})$ and $\mathcal{L}(G') = \mathcal{L}(G)$.
Let $V_\varepsilon = \{A \mid A \to \varepsilon \in R'\}$.

Let R'' be the rule set that is created from R' by removing all rules of the form $A \to \varepsilon$ $(A \neq S)$. Additionally, for every rule of the form $B \to xA$ with $A \in V_\varepsilon, B \in V', x \in \Sigma$ we add a rule $B \to x$ to R''.

Then $G'' = \langle V', \Sigma, R'', S\rangle$ is regular and $\mathcal{L}(G) = \mathcal{L}(G'')$. □

Questions



Questions?

Exercise (Slido)

Consider $G = \langle \{\mathrm{S}, \mathrm{X}, \mathrm{Y}\}, \{\mathtt{a}, \mathtt{b}\}, R, \mathrm{S} \rangle$ with the following rules in R:

$$
\begin{array}{ll}
\mathrm{S} \rightarrow \varepsilon & \mathrm{S} \rightarrow \mathtt{a}\mathrm{X} \\
\mathrm{X} \rightarrow \mathtt{a}\mathrm{X} & \mathrm{X} \rightarrow \mathtt{a}\mathrm{Y} \\
\mathrm{Y} \rightarrow \mathtt{b}\mathrm{Y} & \mathrm{Y} \rightarrow \varepsilon
\end{array}
$$

- Is G a regular grammar?
- Is $\mathcal{L}(G)$ regular?

Summary

Summary

- Regular grammars restrict the usage of ε in rules.
- This restriction is not necessary for the characterization of regular languages but convenient if we want to prove something for all regular languages.