Theorem 1. HamiltonCycle $\leq_p$ TSP.

Proof. We show that function $f(V, E) = \langle S, \texttt{cost}, K \rangle$ with $S = V$, $cost(u, v) = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 2 & \text{otherwise} \end{cases}$ and $K = |V|$ is a polynomial reduction from HamiltonCycle to TSP.

The triple $\langle S, \texttt{cost}, K \rangle$ can be computed in polynomial time: It clearly takes only polynomial time to take over the set V and determine the number of vertices in the graph. For the *cost* function, we can iterate over all pairs of vertices in quadratic time in $|V|$ and set the value for the pair based on a test whether $\{u, v\} \in E$. The exact running time does depend on the representation of E but it is definitively possible in polynomial time in the size of the graph.

We still need to show that $\langle V, E \rangle \in$ HamiltonCycle iff $f(V, E) \in$ TSP.

"$\Rightarrow$" If $\langle V, E \rangle \in$ HamiltonCycle then there is a sequence $\langle v_0, \ldots, v_n \rangle$ of vertices that is a Hamilton cycle of $\langle V, E \rangle$. Since a Hamilton cycle is simple, we know that $\langle v_0, \ldots, v_{n-1} \rangle$ contains each vertex at most once. Since it is Hamiltonian, each vertex is also contained at least once. So overall, $\langle v_0, \ldots, v_{n-1} \rangle$ is a permutation of $V = S$. Since a Hamilton cycle is a path in the graph, we also know that for each $i \in \{0, \ldots, n-1\}$, there is the edge $\{v_i, v_{i+1}\} \in E$, so $cost(v_i, v_{i+1}) = 1$. Therefore $cost(v_{n-1}, v_0) + \sum_{i_0}^{n-2} cost(v_i, v_{i+1}) = |V|$ (for the first term, we use $v_0 = v_n$ from the cyclicity of a Hamilton cycle). We can conclude that $\langle v_0, \ldots, v_{n-1} \rangle$ is a tour for $f(V, E)$ of cost $|V|$ and consequently $f(V, E) \in$ TSP.

"$\Leftarrow$" If $f(V, E) \in$ TSP, let $t = \langle v_1, \ldots, v_n \rangle$ be a tour for $f(V, E)$ of cost at most $K = |V|$. Since t is a permutation of $S = V$, we know that $n = |V|$. Since $cost(x, y) \geq 1$ for all $x, y \in S(= V)$, and $\sum_{i=1}^{n-1} cost(v_i, v_{i+1}) + cost(v_n, v_1) \leq |V|$, we know that the cost between consecutive cities/vertices as well as between v_n and v_1 must be 1. By the definition of the *cost* function, this implies that E contains the corresponding edge. We can conclude that $\pi = \langle v_1, \ldots, v_n, v_1 \rangle$ is a path in $\langle V, E \rangle$. Since the tour t is a permutation of V, we also know that t contains every city/vertex exactly once, so π must be simple and Hamiltonian. Since the first and last component of π is v_1, it also is a cycle and overall, we conclude that π is a Hamilton cycle of $\langle V, E \rangle$ and $\langle V, E \rangle \in$ HamiltonCycle. □