Foundations of Artificial Intelligence

G6. Board Games: Monte-Carlo Tree Search Variants

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Board Games: Overview

chapter overview:

Monte-Carlo Tree Search: Pseudo-Code

function visit_node(n)

if is_terminal(n.position):
    $utility$:= $utility$(n.position)
else:
    s := n.get_unvisited_successor()
    if s **is none**:
        n' := apply_tree_policy(n)
        $utility$:= visit_node(n')
    else:
        $utility$:= simulate_game(s)
        n.add_and_initialize_child_node(s, $utility$)
$n.N := n.N + 1$
$n.\hat{v} := n.\hat{v} + \frac{utility - n.\hat{v}}{n.N}$
return $utility$

Simulation Phase

Simulation Phase

idea: determine initial utility estimate by simulating game following a default policy

Definition (default policy)

Let $\mathcal{S} = \langle S, A, T, s_I, S_G, utility, player \rangle$ be a game.
A default policy for $\mathcal{S}$ is a mapping $\pi_{\text{def}} : S \times A \mapsto [0, 1]$ s.t.

1. $\pi_{\text{def}}(s, a) > 0$ implies that move a is applicable in position s
2. $\sum_{a \in A} \pi_{\text{def}}(s, a) = 1$ for all $s \in S$

In the call to simulate_game(s),

- the default policy is applied starting from position s (determining decisions for both players)
- until a terminal position s_G is reached
- and $utility(s_G)$ is returned.

Implementations

"standard" implementation: Monte-Carlo random walk

- in each position, select a move uniformly at random
- until a terminal position is reached
- policy very cheap to compute
- uninformed $\rightsquigarrow$ often not sufficient for good results
- not always cheap to simulate

alternative: game-specific default policy

- hand-crafted or
- learned offline

Gelly and Silver, Combining Online and Offline Knowledge in UCT (ICML, 2007)

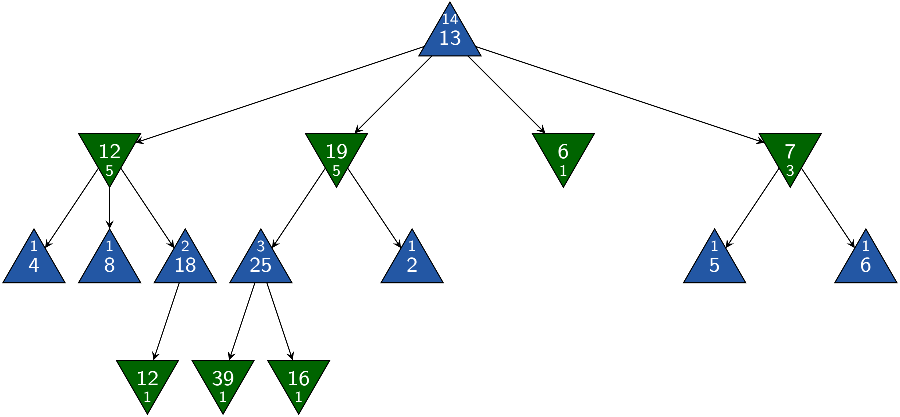
Default Policy vs. Evaluation Function

- default policy simulates a game to obtain utility estimate
  ⇝ default policy must be evaluated in many positions
- if default policy is expensive to compute or poorly informed, simulations are expensive
- observe: simulating a game to the end is just a specific implementation of an evaluation function
- many modern implementations replace default policy with evaluation function that directly computes a utility estimate

⇝ MCTS becomes a heuristic search algorithm

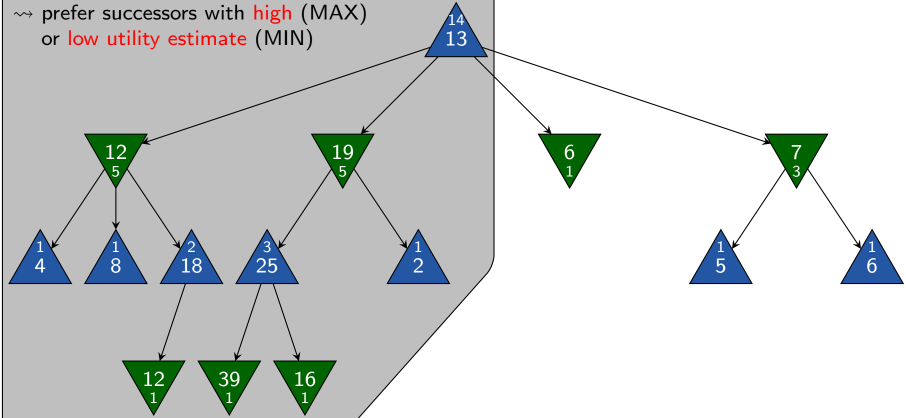
Tree Policy

Objective of Tree Policy (1)



Objective of Tree Policy (1)

exploit collected information to
focus search on promising areas
⇝ prefer successors with high (MAX)
or low utility estimate (MIN)



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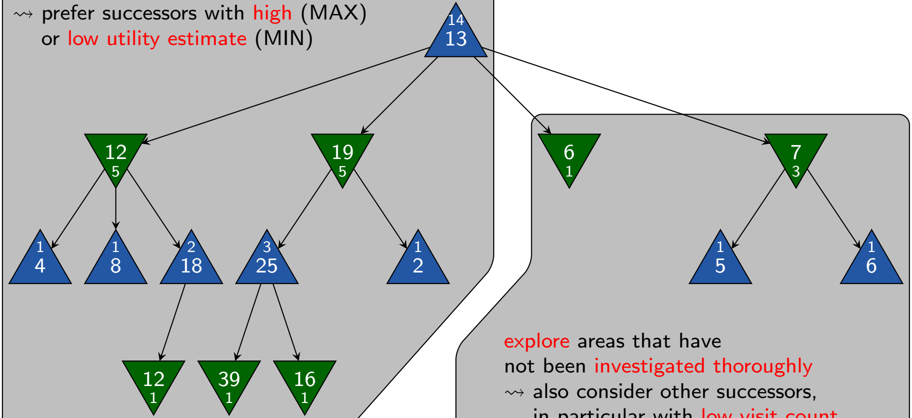

explore areas that have not been investigated thoroughly

⇝ also consider other successors, in particular with low visit count

Objective of Tree Policy (1)

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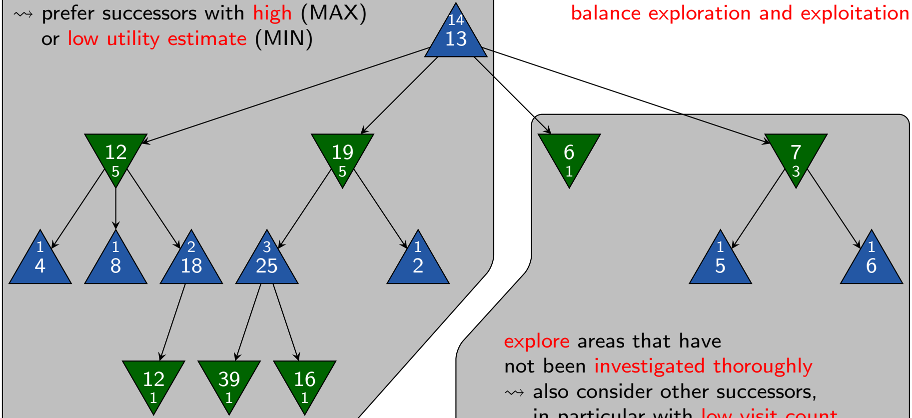
These are contradictory objectives!
⇝ 1st central challenge for tree policy:
balance exploration and exploitation



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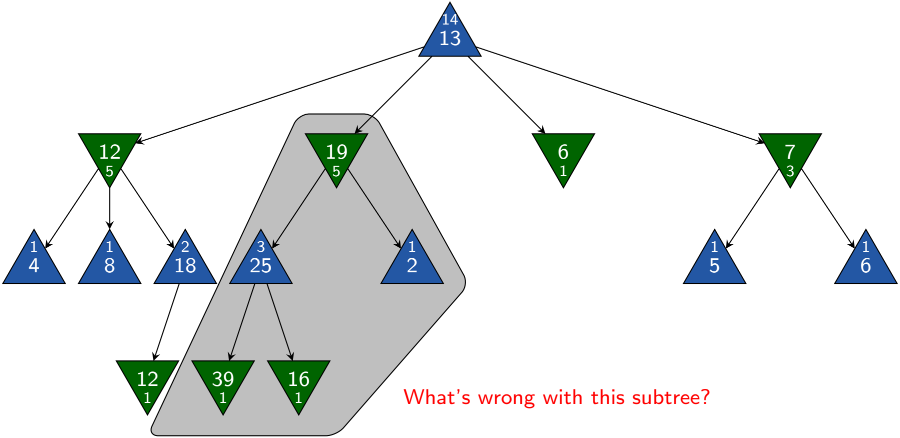
Objective of Tree Policy (2)



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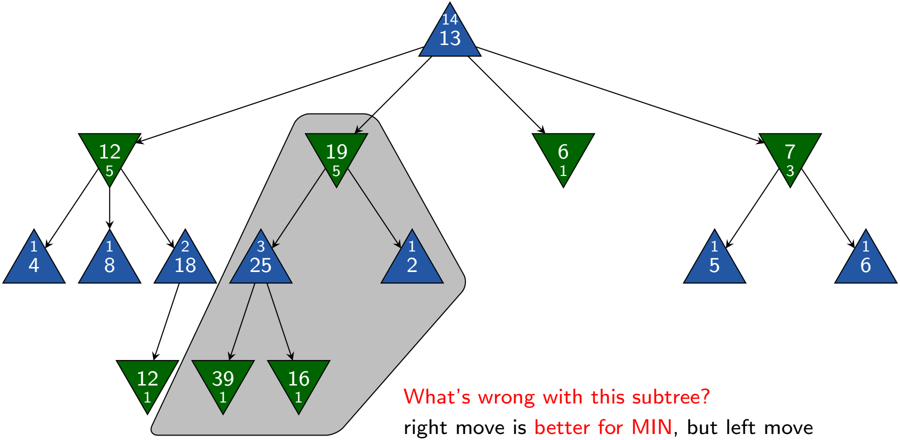
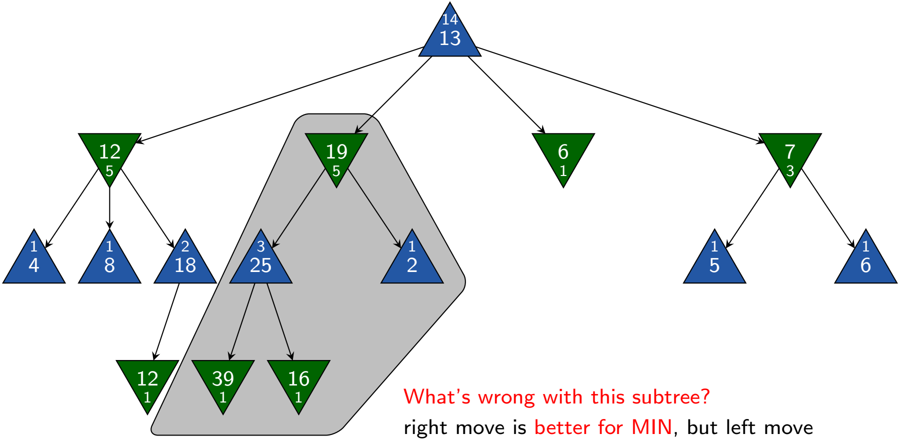


What's wrong with this subtree?

right move is better for MIN, but left move has higher influence on utility estimate

Objective of Tree Policy (2)



What's wrong with this subtree?

right move is better for MIN, but left move has higher influence on utility estimate

⇝ 2nd central challenge for tree policy: exploit much more often than explore (in the limit)

Asymptotic Optimality

Definition (asymptotic optimality)

Let $\mathcal{S}$ be a game with set of positions S.
Let $v^*(s)$ denote the (true) utility of position $s \in S$.

Let $n.\hat{v}^k$ denote the utility estimate of a search node n after k trials.

An MCTS algorithm is asymptotically optimal if

$$\lim_{k \to \infty} n.\hat{v}^k = v^*(n.\text{position})$$

for all search nodes n.

Asymptotic Optimality

a tree policy is asymptotically optimal if

- it explores forever:
  - every position is eventually added to the game tree and visited infinitely often (requires that the game tree is finite)
  - ⇝ after a finite number of trials, all trials end in a terminal position and the default policy is no longer used
- and it is greedy in the limit:
  - the probability that an optimal move is selected converges to 1
  - ⇝ in the limit, backups based on trials where only an optimal policy is followed dominate suboptimal backups

Tree Policy: Examples

ε-greedy: Idea and Example

- tree policy with constant parameter ε
- with probability $1 - \varepsilon$, pick a greedy move which leads to:
  - a successor with highest utility estimate (for MAX)
  - a successor with lowest utility estimate (for MIN)
- otherwise, pick a non-greedy successor uniformly at random

ε-greedy: Idea and Example

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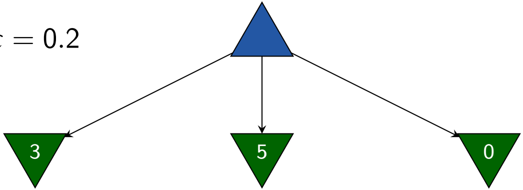
$\varepsilon = 0.2$

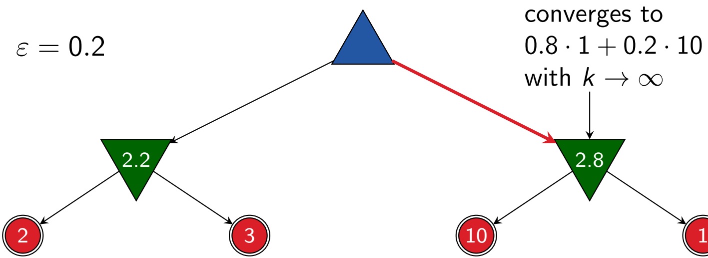


$\mathbb{P}(n_1) = 0.1$ $\mathbb{P}(n_2) = 0.8$ $\mathbb{P}(n_3) = 0.1$

($\mathbb{P}(n)$ denotes probability that successor n is selected)

ε-greedy: Optimality

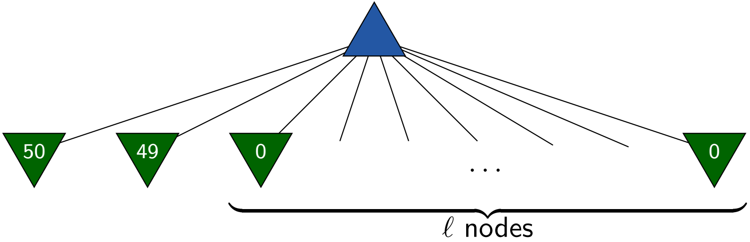
ε-greedy is not asymptotically optimal:



variants that are asymptotically optimal exist
(e.g., decaying ε, minimax backups)

ε-greedy: Weakness

problem:
when ε-greedy explores, all non-greedy moves are treated equally



e.g., $\varepsilon = 0.2, \ell = 9$: $\mathbb{P}(n_1) = 0.8$, $\mathbb{P}(n_2) = 0.02$

Softmax: Idea and Example

- tree policy with constant parameter $\tau > 0$
- select moves with a frequency that directly relates to their utility estimate
- Boltzmann exploration selects moves proportionally to $\mathbb{P}(n) \propto e^{\frac{n.\hat{v}}{\tau}}$ for MAX and to $\mathbb{P}(n) \propto e^{\frac{-n.\hat{v}}{\tau}}$ for MIN

Softmax: Idea and Example

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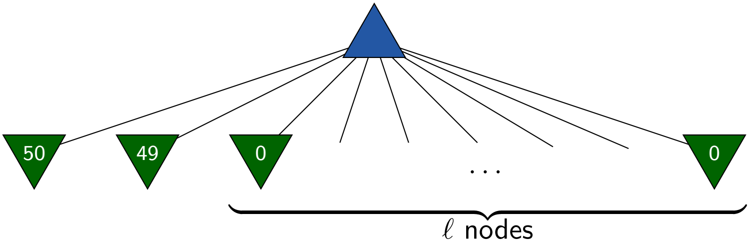
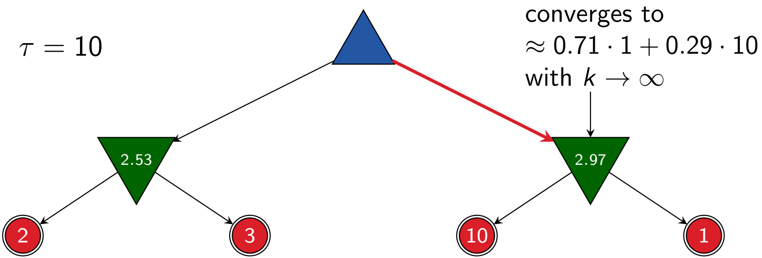


e.g., $\tau = 10, \ell = 9$: $\mathbb{P}(n_1) \approx 0.51$, $\mathbb{P}(n_2) \approx 0.46$

Boltzmann exploration: Optimality

Boltzmann exploration is not asymptotically optimal:



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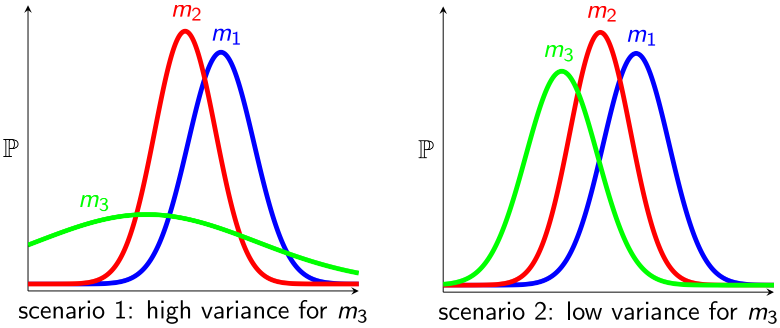
Boltzmann Exploration: Weakness



- Boltzmann exploration only considers mean of sampled utilities for the given moves
- as we sample the same node many times, we can also gather information about variance (how reliable the information is)
- Boltzmann exploration ignores the variance, treating the two scenarios equally

Upper Confidence Bounds: Idea

balance exploration and exploitation by preferring moves that

- have been successful in earlier iterations (exploit)
- have been selected rarely (explore)

Upper Confidence Bounds: Idea

upper confidence bound for MAX:

- select successor n' of n that maximizes $n'.\hat{v} + B(n')$
- based on utility estimate $n'.\hat{v}$
- and a bonus term $B(n')$
- select $B(n')$ such that $v^*(n'.\text{position}) \leq n'.\hat{v} + B(n')$ with high probability
- idea: $n'.\hat{v} + B(n')$ is an upper confidence bound on $n'.\hat{v}$ under the collected information

(for MIN: maximize $-n'.\hat{v} + B(n')$)

Upper Confidence Bounds: UCB1

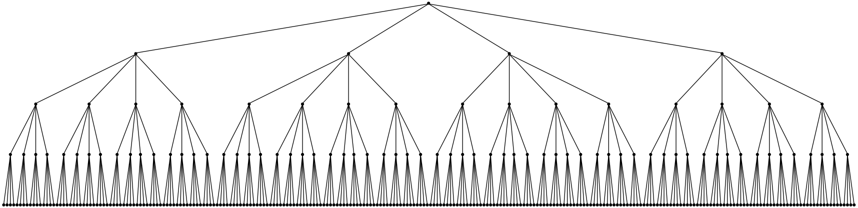
- use $B(n') = \sqrt{\frac{2 \cdot \ln n.N}{n'.N}}$ as bonus term
- bonus term is derived from Chernoff-Hoeffding bound, which
  - gives the probability that a sampled value (here: $n'.\hat{v}$)
  - is far from its true expected value (here: $v^*(n'.\text{position})$)
  - in dependence of the number of samples (here: $n'.N$)
- picks an optimal move exponentially more often in the limit

UCB1 is asymptotically optimal.

Comparison of Game Algorithms

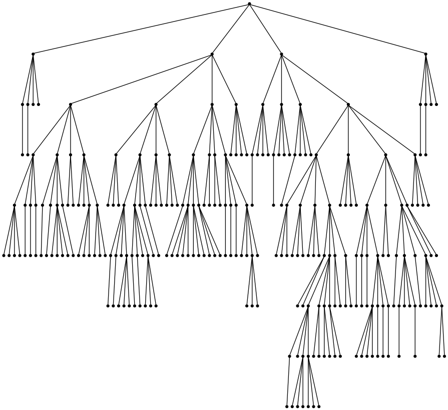
Minimax Tree

full tree up to depth 4



alpha-beta search with same effort:
⇝ depth 6–8 with good move ordering

MCTS Tree



Summary

Summary

- tree policy is crucial for MCTS
  - ϵ-greedy favors greedy moves and treats all others equally
  - Boltzmann exploration selects moves proportionally to an exponential function of their utility estimates
  - UCB1 favors moves that were successful in the past or have been explored rarely
- for each, there are applications where they perform best
- good default policies are domain-dependent and hand-crafted or learned offline
- using evaluation functions instead of a default policy often pays off