Foundations of Artificial Intelligence

G6. Board Games: Monte-Carlo Tree Search Variants

Malte Helmert

University of Basel

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Board Games: Overview

chapter overview:

- G1. Introduction and State of the Art
- G2. Minimax Search and Evaluation Functions
- G3. Alpha-Beta Search
- G4. Stochastic Games
- G5. Monte-Carlo Tree Search Framework
- G6. Monte-Carlo Tree Search Variants

Monte-Carlo Tree Search: Pseudo-Code

function visit_node(n)

if is_terminal(n.position):
  $utility := utility(n.\text{position})$
else:
  $s := n$.get_unvisited_successor()
  if s **is none**:
    $n' :=$ apply_tree_policy(n)
    $utility :=$ visit_node(n')
  else:
    $utility :=$ simulate_game(s)
    n.add_and_initialize_child_node(s, $utility$)
$n.N := n.N + 1$
$n.\hat{v} := n.\hat{v} + \frac{utility - n.\hat{v}}{n.N}$
return $utility$

G6.1 Simulation Phase

Simulation Phase

idea: determine initial utility estimate by simulating game following a default policy

Definition (default policy)

Let $\mathcal{S} = \langle S, A, T, s_I, S_G, \mathit{utility}, \mathit{player} \rangle$ be a game.
A default policy for $\mathcal{S}$ is a mapping $\pi_{\mathrm{def}} : S \times A \mapsto [0, 1]$ s.t.

1. $\pi_{\mathrm{def}}(s, a) > 0$ implies that move a is applicable in position s
2. $\sum_{a \in A} \pi_{\mathrm{def}}(s, a) = 1$ for all $s \in S$

In the call to simulate_game(s),

- the default policy is applied starting from position s (determining decisions for both players)
- until a terminal position s_G is reached
- and $\mathit{utility}(s_G)$ is returned.

Implementations

"standard" implementation: Monte-Carlo random walk

- in each position, select a move uniformly at random
- until a terminal position is reached
- policy very cheap to compute
- uninformed $\rightsquigarrow$ often not sufficient for good results
- not always cheap to simulate

alternative: game-specific default policy

- hand-crafted or
- learned offline

Gelly and Silver, Combining Online and Offline Knowledge in UCT (ICML, 2007)

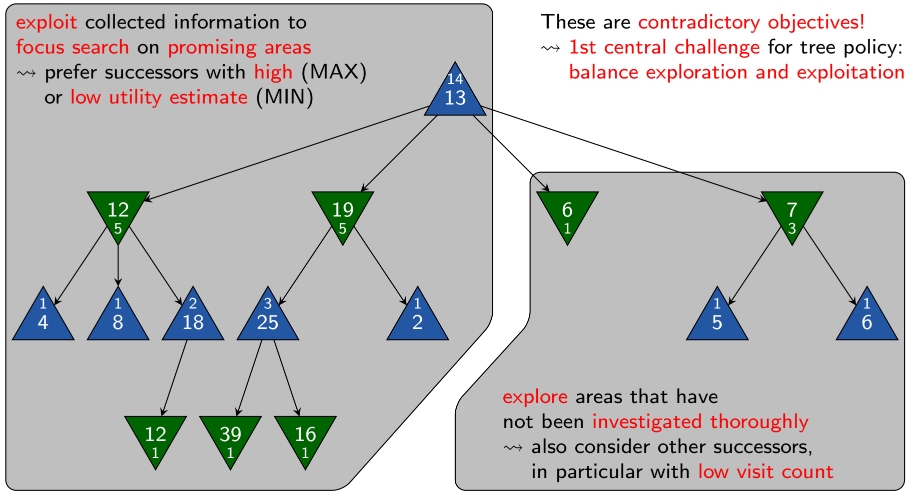
Default Policy vs. Evaluation Function

- default policy simulates a game to obtain utility estimate $\rightsquigarrow$ default policy must be evaluated in many positions
- if default policy is expensive to compute or poorly informed, simulations are expensive
- observe: simulating a game to the end is just a specific implementation of an evaluation function
- many modern implementations replace default policy with evaluation function that directly computes a utility estimate

$\rightsquigarrow$ MCTS becomes a heuristic search algorithm

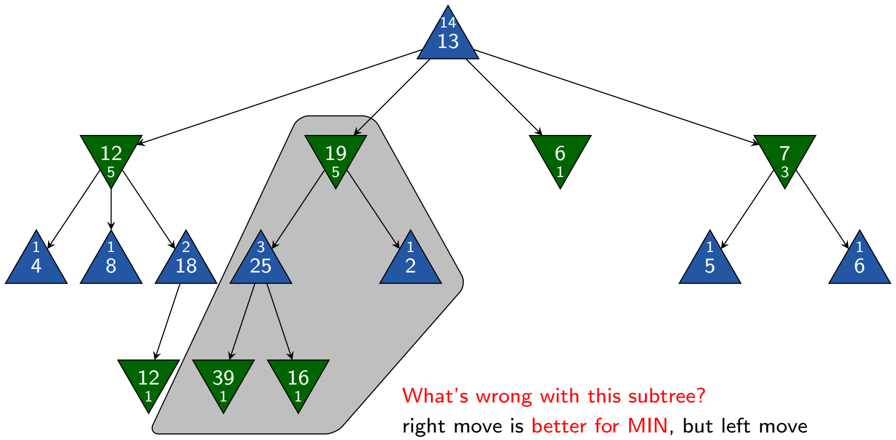
G6.2 Tree Policy

Objective of Tree Policy (1)



These are contradictory objectives!
⇝ 1st central challenge for tree policy:
balance exploration and exploitation

Objective of Tree Policy (2)



What's wrong with this subtree?
right move is better for MIN, but left move
has higher influence on utility estimate
⇝ 2nd central challenge for tree policy:
exploit much more often than explore
(in the limit)

Asymptotic Optimality

Definition (asymptotic optimality)

Let $\mathcal{S}$ be a game with set of positions S.
Let $v^*(s)$ denote the (true) utility of position $s \in S$.

Let $n.\hat{v}^k$ denote the utility estimate
of a search node n after k trials.

An MCTS algorithm is asymptotically optimal if

$$\lim_{k\to\infty} n.\hat{v}^k = v^*(n.\text{position})$$

for all search nodes n.

Asymptotic Optimality

a tree policy is asymptotically optimal if

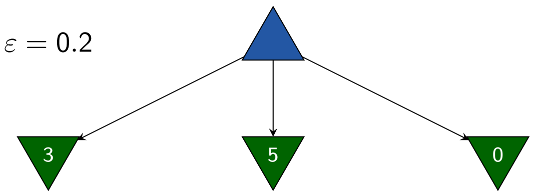
- it explores forever:
  - every position is eventually added to the game tree and visited infinitely often (requires that the game tree is finite)
  - $\rightsquigarrow$ after a finite number of trials, all trials end in a terminal position and the default policy is no longer used
- and it is greedy in the limit:
  - the probability that an optimal move is selected converges to 1
  - $\rightsquigarrow$ in the limit, backups based on trials where only an optimal policy is followed dominate suboptimal backups

G6.3 Tree Policy: Examples

ε-greedy: Idea and Example

- tree policy with constant parameter ε
- with probability $1 - \varepsilon$, pick a greedy move which leads to:
  - a successor with highest utility estimate (for MAX)
  - a successor with lowest utility estimate (for MIN)
- otherwise, pick a non-greedy successor uniformly at random

$\varepsilon = 0.2$



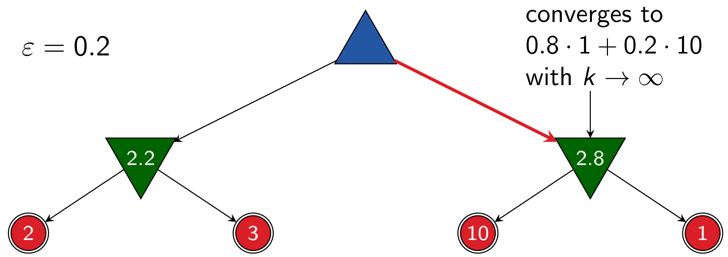
$\mathbb{P}(n_1) = 0.1$ $\mathbb{P}(n_2) = 0.8$ $\mathbb{P}(n_3) = 0.1$

($\mathbb{P}(n)$ denotes probability that successor n is selected)

ε-greedy: Optimality

ε-greedy is not asymptotically optimal:

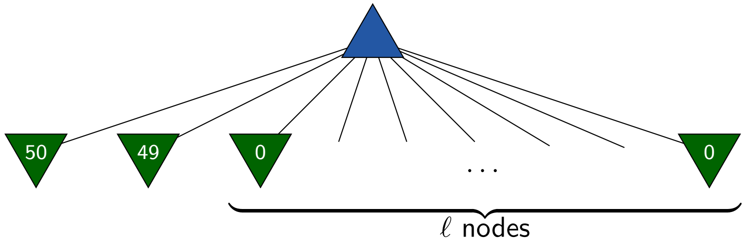
$\varepsilon = 0.2$



variants that are asymptotically optimal exist (e.g., decaying ε, minimax backups)

ε-greedy: Weakness

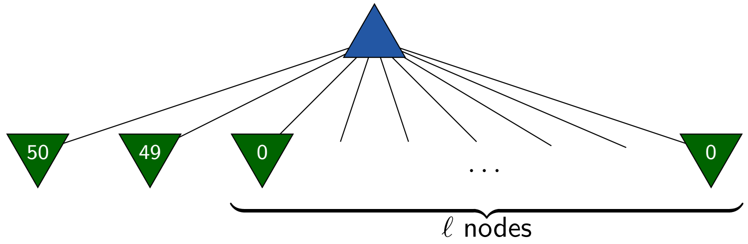
problem:
when ε-greedy explores, all non-greedy moves are treated equally



e.g., $\varepsilon = 0.2, \ell = 9$: $\mathbb{P}(n_1) = 0.8$, $\mathbb{P}(n_2) = 0.02$

Softmax: Idea and Example

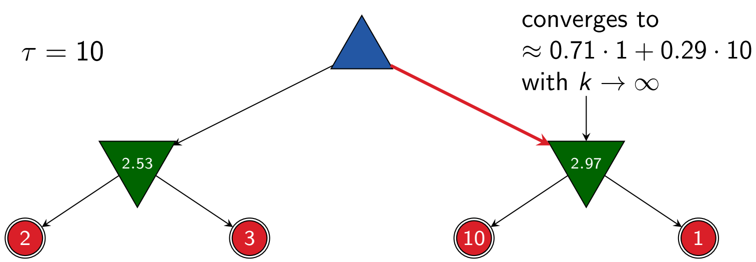
- tree policy with constant parameter $\tau > 0$
- select moves with a frequency that directly relates to their utility estimate
- Boltzmann exploration selects moves proportionally to $\mathbb{P}(n) \propto e^{\frac{n.\hat{v}}{\tau}}$ for MAX and to $\mathbb{P}(n) \propto e^{\frac{-n.\hat{v}}{\tau}}$ for MIN



e.g., $\tau = 10, \ell = 9$: $\mathbb{P}(n_1) \approx 0.51$, $\mathbb{P}(n_2) \approx 0.46$

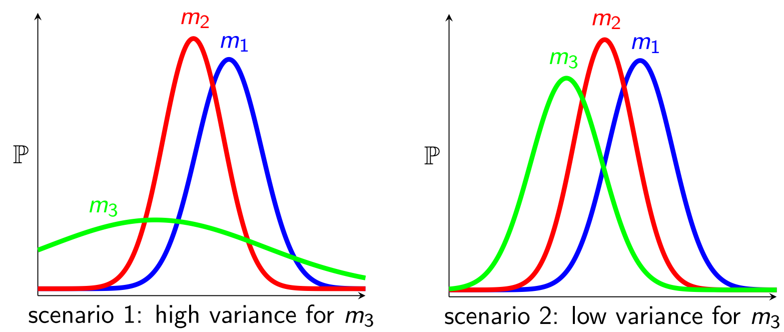
Boltzmann exploration: Optimality

Boltzmann exploration is not asymptotically optimal:



variants that are asymptotically optimal exist
(e.g., decaying τ, minimax backups)

Boltzmann Exploration: Weakness



- Boltzmann exploration only considers mean of sampled utilities for the given moves
- as we sample the same node many times, we can also gather information about variance (how reliable the information is)
- Boltzmann exploration ignores the variance, treating the two scenarios equally

Upper Confidence Bounds: Idea

balance exploration and exploitation by preferring moves that

- have been successful in earlier iterations (exploit)
- have been selected rarely (explore)

Upper Confidence Bounds: Idea

upper confidence bound for MAX:

- select successor n' of n that maximizes $n'.\hat{v} + B(n')$
- based on utility estimate $n'.\hat{v}$
- and a bonus term $B(n')$
- select $B(n')$ such that $v^*(n'.\text{position}) \leq n'.\hat{v} + B(n')$ with high probability
- idea: $n'.\hat{v} + B(n')$ is an upper confidence bound on $n'.\hat{v}$ under the collected information

(for MIN: maximize $-n'.\hat{v} + B(n')$)

Upper Confidence Bounds: UCB1

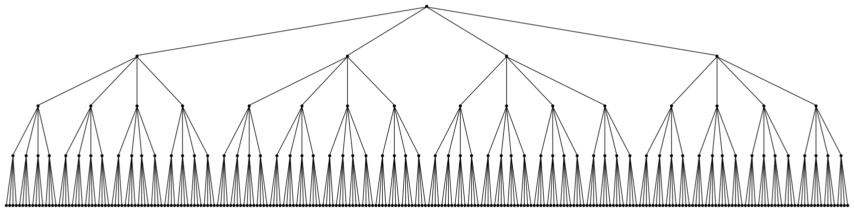
- use $B(n') = \sqrt{\frac{2 \cdot \ln n.N}{n'.N}}$ as bonus term
- bonus term is derived from Chernoff-Hoeffding bound, which
  - gives the probability that a sampled value (here: $n'.\hat{v}$)
  - is far from its true expected value (here: $v^*(n'.\text{position})$)
  - in dependence of the number of samples (here: $n'.N$)
- picks an optimal move exponentially more often in the limit

UCB1 is asymptotically optimal.

G6.4 Comparison of Game Algorithms

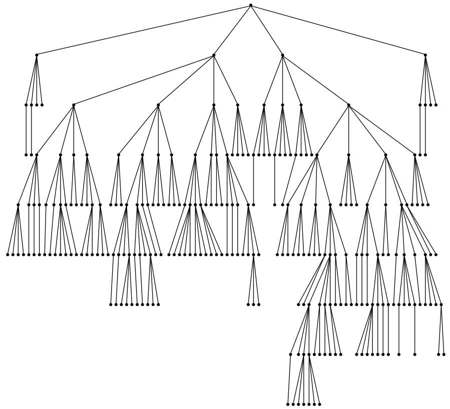
Minimax Tree

full tree up to depth 4



alpha-beta search with same effort:
$\rightsquigarrow$ depth 6–8 with good move ordering

MCTS Tree



G6.5 Summary

Summary

- tree policy is crucial for MCTS
  - ϵ-greedy favors greedy moves and treats all others equally
  - Boltzmann exploration selects moves proportionally to an exponential function of their utility estimates
  - UCB1 favors moves that were successful in the past or have been explored rarely
- for each, there are applications where they perform best
- good default policies are domain-dependent and hand-crafted or learned offline
- using evaluation functions instead of a default policy often pays off