Foundations of Artificial Intelligence

G4. Board Games: Stochastic Games

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May 19, 2025

Board Games: Overview

chapter overview:

Expected Value

Discrete Random Variable

- a random event (like the result of a die roll)
  - is described in terms of a random variable X
  - with associated domain $\mathrm{dom}(X)$
  - and a probability distribution over the domain
- if the number of outcomes of a random event is finite (like here), the random variable is a discrete random variable
- and the probability distribution is given as a probability $P(X = x)$ that the outcome is $x \in \mathrm{dom}(X)$

Discrete Random Variable: Example

informal description:

- you plan to invest in stocks and can afford one share
- your analyst expects these stock price changes:

Bellman Inc.	Markov Tec.
+2 with 30%	+4 with 20%
+1 with 60%	+2 with 30%
±0 with 10%	−1 with 50%

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± 0 with 10%	-1 with 50%

formal model:

- discrete random variables B and M
- $\mathrm{dom}(B) = \{2, 1, 0\}$
  $\mathrm{dom}(M) = \{4, 2, -1\}$
- $P(B = 2) = 0.3 \quad P(M = 4) = 0.2$
  $P(B = 1) = 0.6 \quad P(M = 2) = 0.3$
  $P(B = 0) = 0.1 \quad P(M = -1) = 0.5$

Expected Value

- the expected value $\mathbb{E}[X]$ of a random variable X is a weighted average of its outcomes
- it is computed as the probability-weighted sum of all outcomes $x \in \mathrm{dom}(X)$, i.e.,

$$\mathbb{E}[X] = \sum_{x \in \mathrm{dom}(X)} P(X = x) \cdot x$$

- in stochastic environments, it is rational to deal with uncertainty by optimizing expected values

Expected Value: Example

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expected gain:

$$\begin{aligned}\mathbb{E}[B] &= P(B = 2) \cdot 2 + P(B = 1) \cdot 1 + P(B = 0) \cdot 0 \\ &= 0.3 \cdot 2 + 0.6 \cdot 1 + 0.1 \cdot 0 = 1.2\end{aligned}$$

$$\begin{aligned}\mathbb{E}[M] &= P(M = 4) \cdot 4 + P(M = 2) \cdot 2 + P(M = -1) \cdot -1 \\ &= 0.2 \cdot 4 + 0.3 \cdot 2 + 0.5 \cdot -1 = 0.9\end{aligned}$$

Expected Value: Example

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expected gain:

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$$\mathbb{E}[M] = P(M = 4) \cdot 4 + P(M = 2) \cdot 2 + P(M = -1) \cdot -1 = 0.2 \cdot 4 + 0.3 \cdot 2 + 0.5 \cdot -1 = 0.9$$

rational decision: buy Bellman Inc.

Stochastic Games

Definition

Definition (stochastic game)

A stochastic game is a 7-tuple $\mathcal{S} = \langle S, A, T, s_\mathrm{I}, S_\mathrm{G}, \mathit{utility}, \mathit{player} \rangle$ with

- finite set of positions S
- finite set of moves A
- transition function $T : S \times A \times S \mapsto [0, 1]$ that is well-defined for $\langle s, a \rangle$ (see below)
- initial position $s_\mathrm{I} \in S$
- set of terminal positions $S_\mathrm{G} \subseteq S$
- utility function $\mathit{utility} : S_\mathrm{G} \to \mathbb{R}$
- player function $\mathit{player} : S \setminus S_\mathrm{G} \to \{\mathrm{MAX}, \mathrm{MIN}\}$

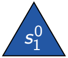
A transition function is well-defined for $\langle s, a \rangle$ if $\sum_{s' \in S} T(s, a, s') = 1$ (then a is applicable in s) or $\sum_{s' \in S} T(s, a, s') = 0$.

Example: Stochastic Inc-and-Square Game

- As an example, we consider a variant of the bounded inc-and-square game from Chapter G1.
- The sqr move now acts stochastically:
  - It squares the current value v (mod 10) with probability $\frac{v}{10}$.
  - Otherwise it doubles the current value v (mod 10) (with prob. $1 - \frac{v}{10}$).
- We also reduce the maximum game length to 3 moves (counting both players) to make the example smaller.
- Everything else stays the same.

Expectiminimax

Idea and Example



- depth-first search in game tree
- determine utility value of terminal positions with utility function
- compute utility value of inner nodes bottom-up through the tree:
  - MIN's turn: utility value is minimum of utility values of children
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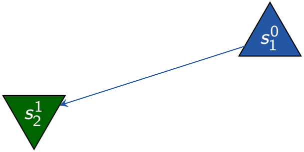

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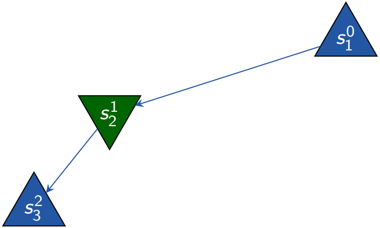

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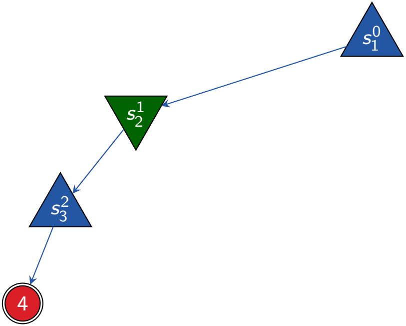

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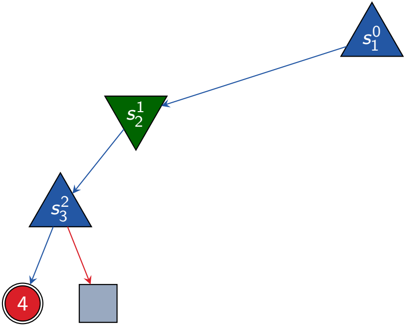

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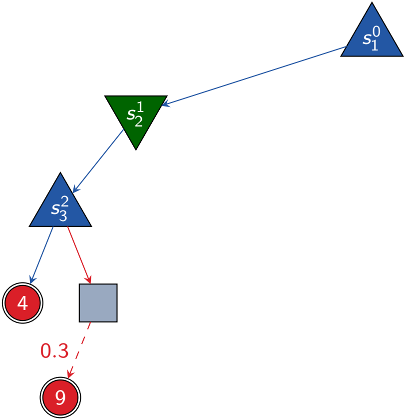

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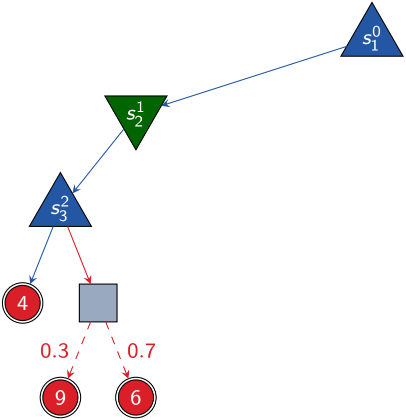

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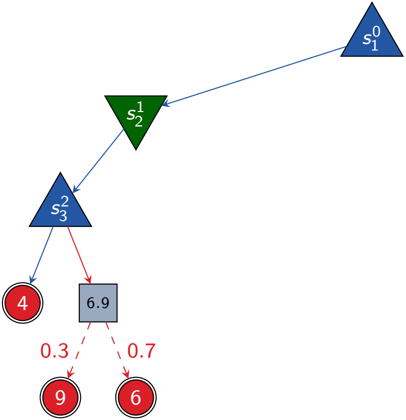

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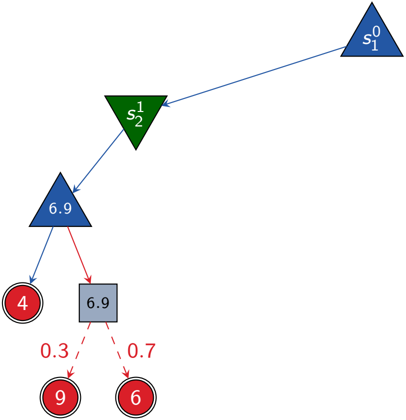

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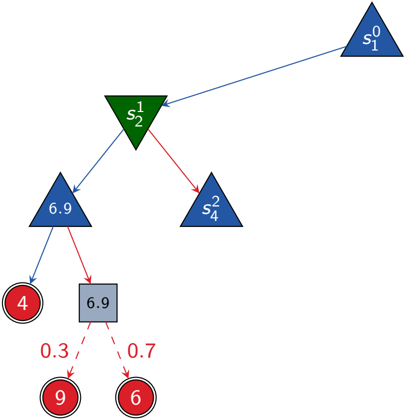

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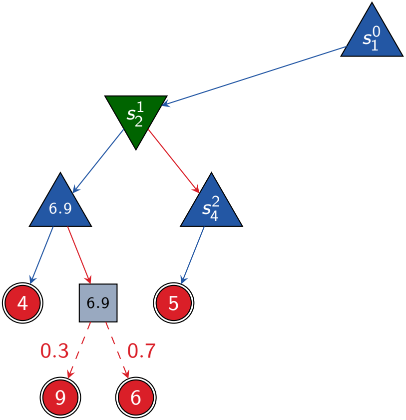

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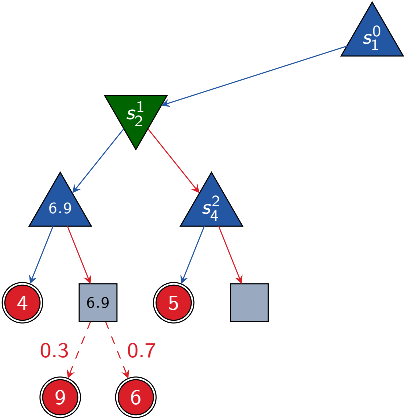

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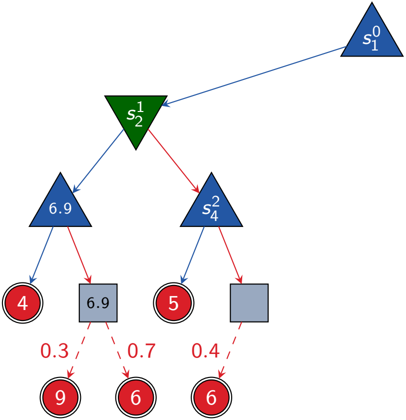

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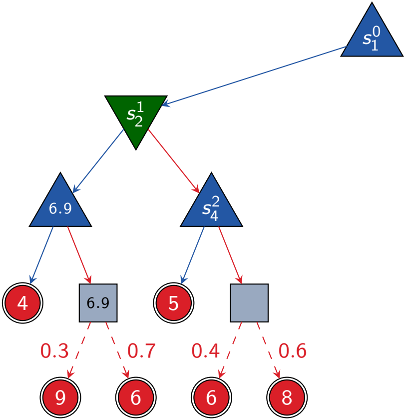

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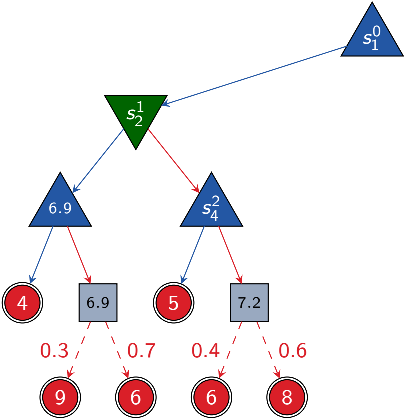

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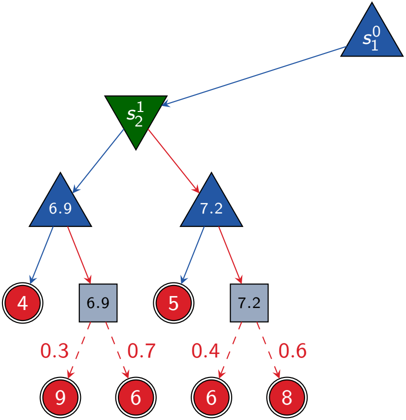

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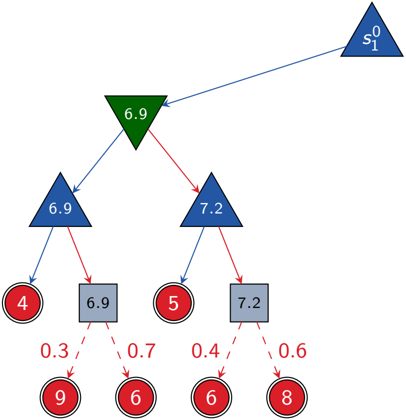

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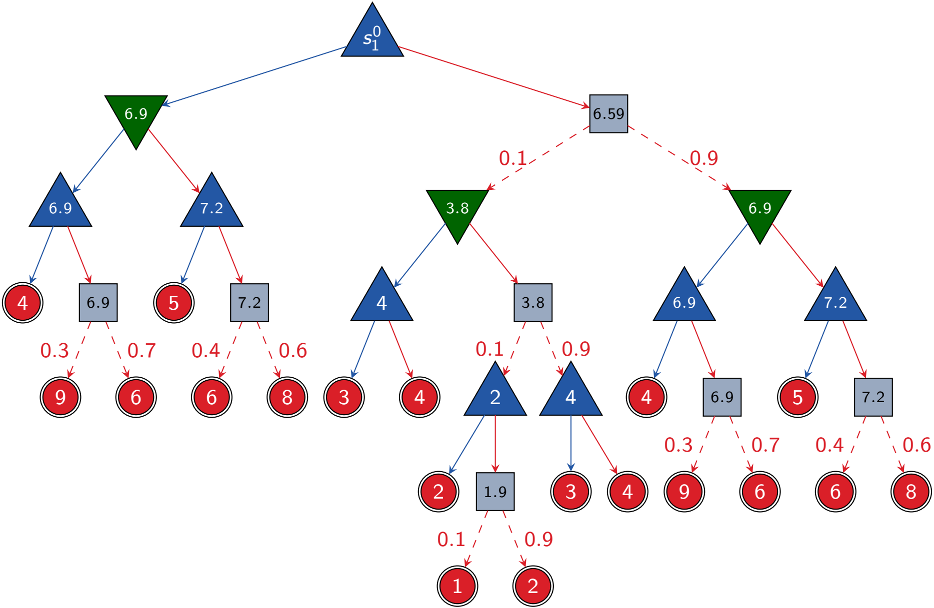

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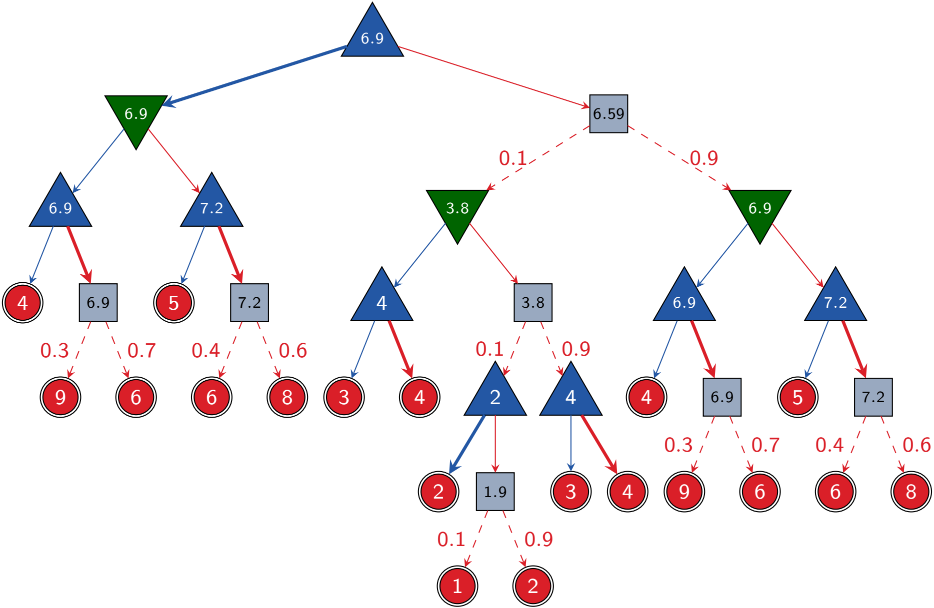

Discussion

- expectiminimax is the simplest (decent) search algorithm for stochastic games
- yields optimal policy (in the game-theoretic sense, i.e., under the assumption that the opponent plays perfectly)
- MAX obtains at least the utility value computed for the root in expectation, no matter how MIN plays
- if MIN plays perfectly, MAX obtains exactly the computed value in expectation

The same improvements as for minimax are possible (evaluation functions, alpha-beta search).

Summary

Summary

- Stochastic games are board games with an additional element of chance.
- Expectiminimax is a minimax variant for stochastic games with identical behavior in MAX and MIN nodes.
- In chance nodes, it propagates the expected value (probability-weighted sum) of all successors.
- Expectiminimax has same guarantees as minimax, but in expectation.