Foundations of Artificial Intelligence

G3. Board Games: Alpha-Beta Search

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Board Games: Overview

chapter overview:

- G1. Introduction and State of the Art
- G2. Minimax Search and Evaluation Functions
- G3. Alpha-Beta Search
- G4. Stochastic Games
- G5. Monte-Carlo Tree Search Framework
- G6. Monte-Carlo Tree Search Configurations

Limitations of Minimax

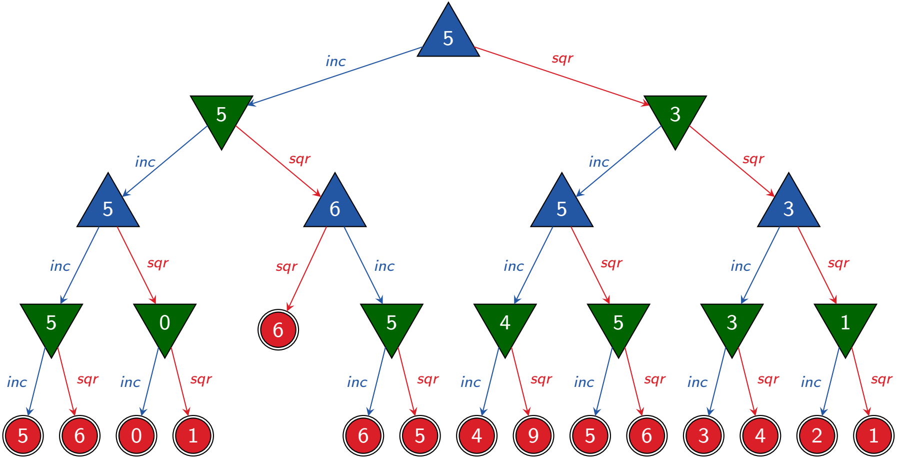
What if the size of the game tree is too big for minimax?

$\rightsquigarrow$ heuristic alpha-beta search

- heuristics (evaluation functions): previous chapter
- alpha-beta search: this chapter

Alpha-Beta Search

Can We Save Search Effort?



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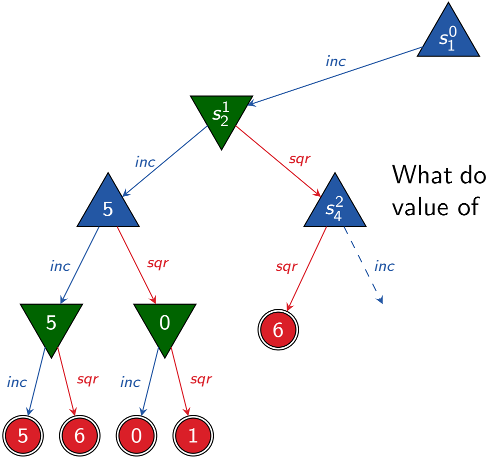

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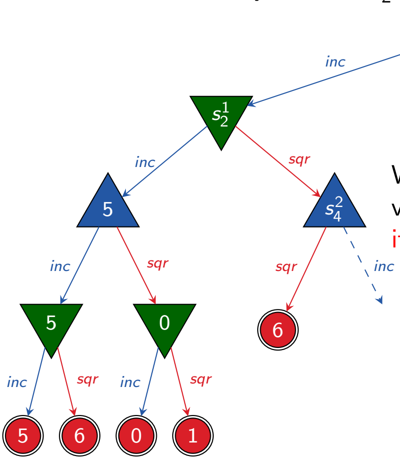
Can We Save Search Effort?

And about the utility value of s_2^1?



What do we know about the utility value of s_4^2 in this situation?

it's 6 or higher

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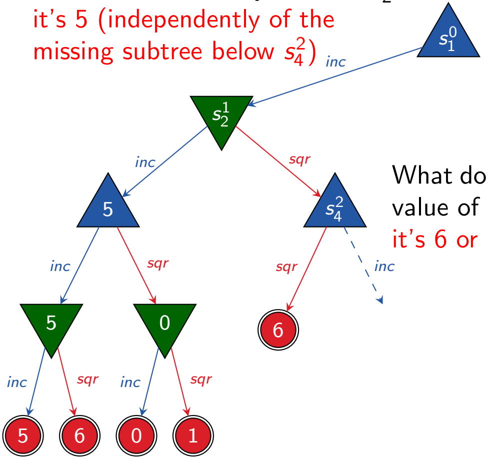
And about the utility value of s_2^1?
it's 5 (independently of the missing subtree below s_4^2)



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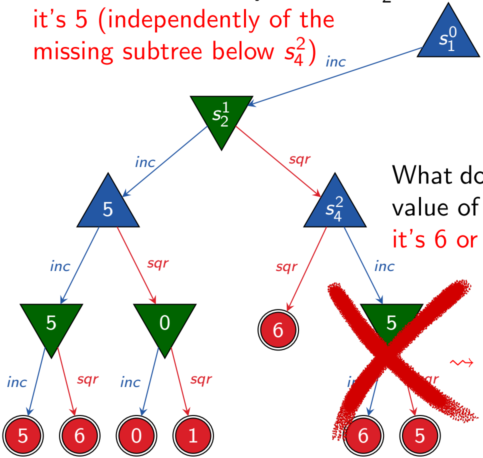


What do we know about the utility value of s_4^2 in this situation?

it's 6 or higher

⇝ we don't have to look at this

Idea

idea: for every search node, use two values α and β such that we know that the subtree rooted at the node

- is irrelevant if its utility is $\leq \alpha$
  because MAX will prevent entering it when playing optimally
- is irrelevant if its utility is $\geq \beta$
  because MIN will prevent entering it when playing optimally

We can prune every node with $\alpha \geq \beta$
because it must be irrelevant (no matter what its utility is).

Alpha-Beta Search: Pseudo Code

- algorithm skeleton the same as minimax
- function signature extended by two variables α and β

function alpha-beta-main(p)

$\langle v, move \rangle :=$ alpha-beta($p, -\infty, +\infty$)

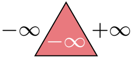
return *move*

Alpha-Beta Search: Pseudo-Code

function alpha-beta(p, α, β)

if p is terminal position:
    return $\langle$*utility*(p), **none**$\rangle$
initialize v and *best_move* [as in minimax]
for each $\langle$*move*, $p'\rangle \in$ succ(p):
    $\langle v'$, *best_move*$'\rangle :=$ *alpha-beta*(p', α, β)
    update v and *best_move* [as in minimax]
    if *player*(p) = MAX:
        if $v \geq \beta$:
            return $\langle v$, **none**$\rangle$
        $\alpha := \max\{\alpha, v\}$
    if *player*(p) = MIN:
        if $v \leq \alpha$:
            return $\langle v$, **none**$\rangle$
        $\beta := \min\{\beta, v\}$
return $\langle v$, *best_move*$\rangle$

Example



- α: lower bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
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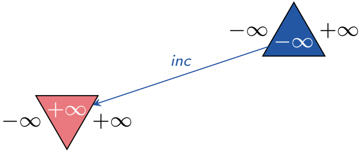

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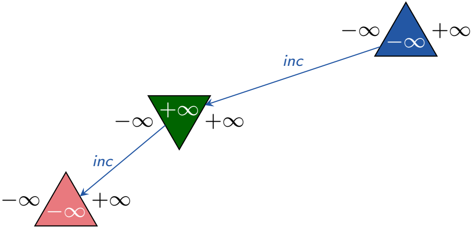

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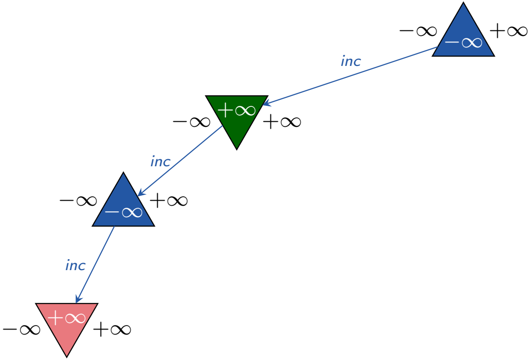

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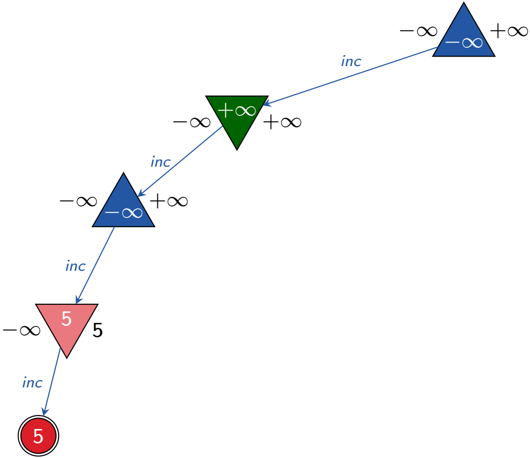

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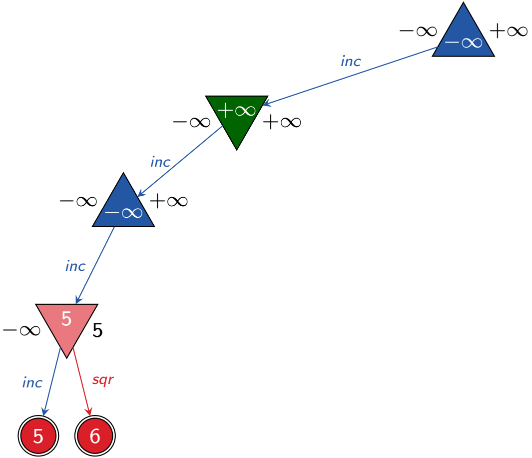

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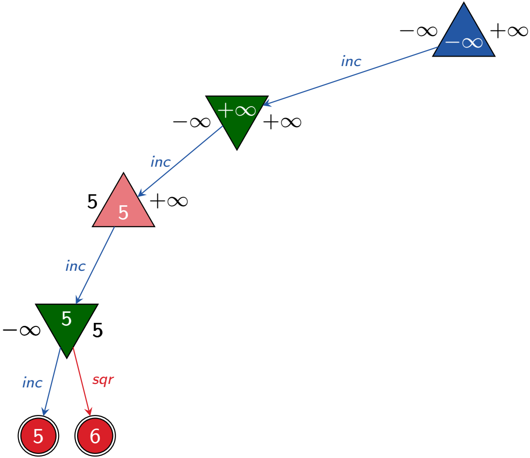

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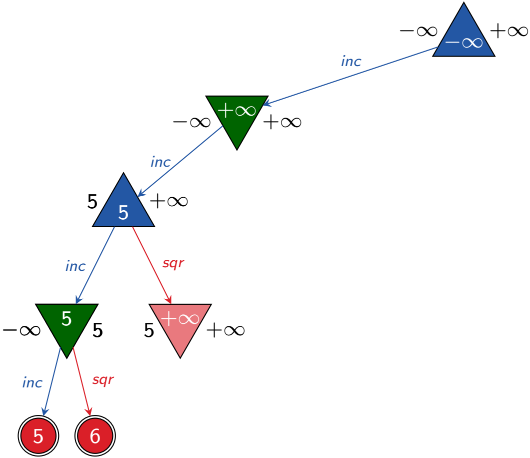

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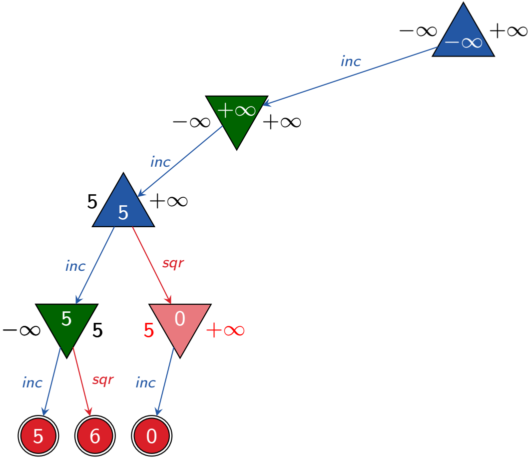

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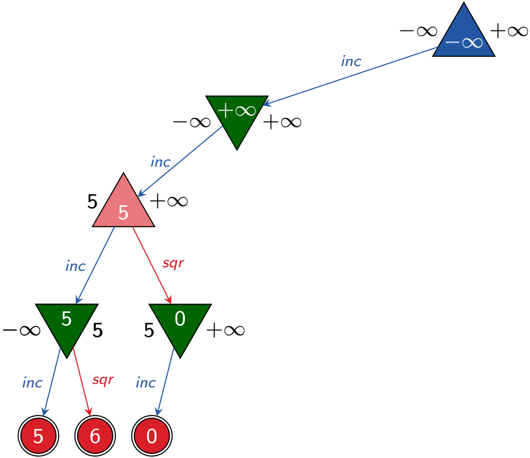

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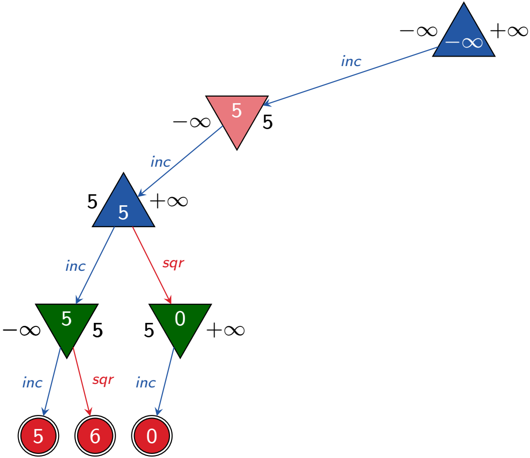

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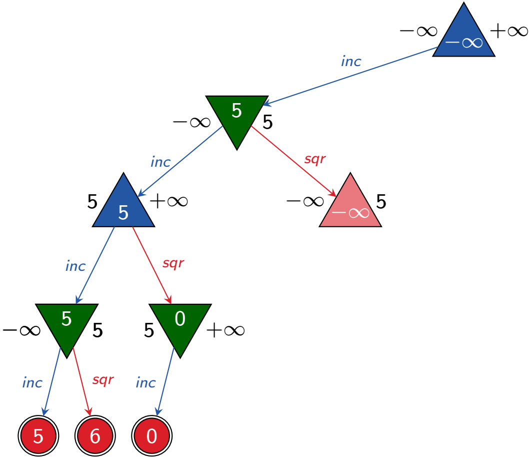

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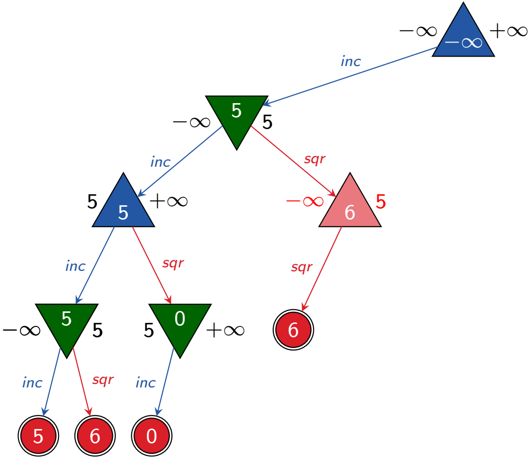

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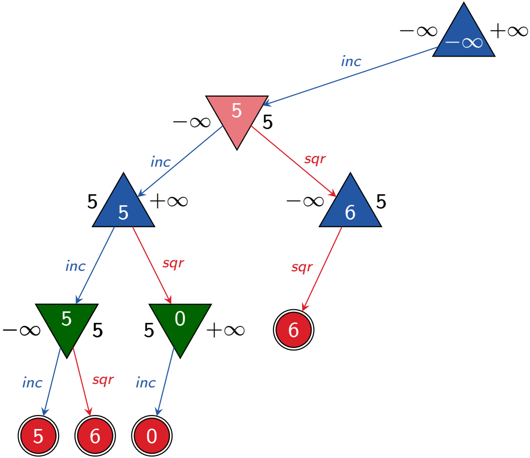

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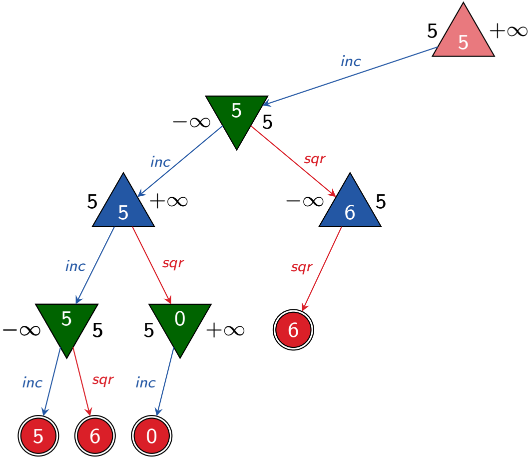

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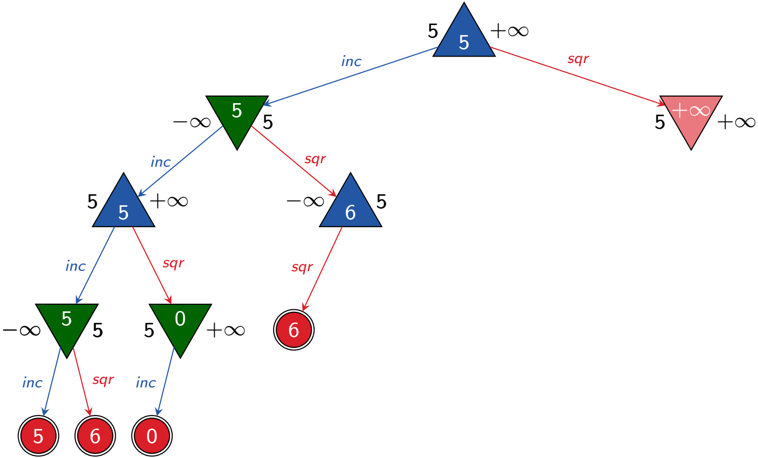

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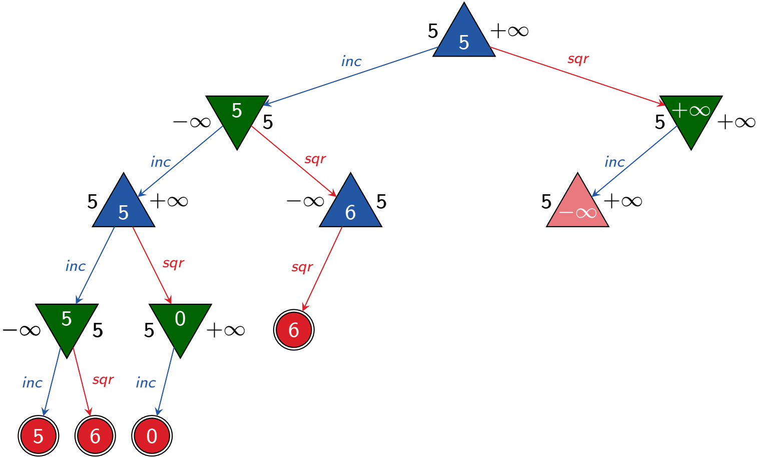

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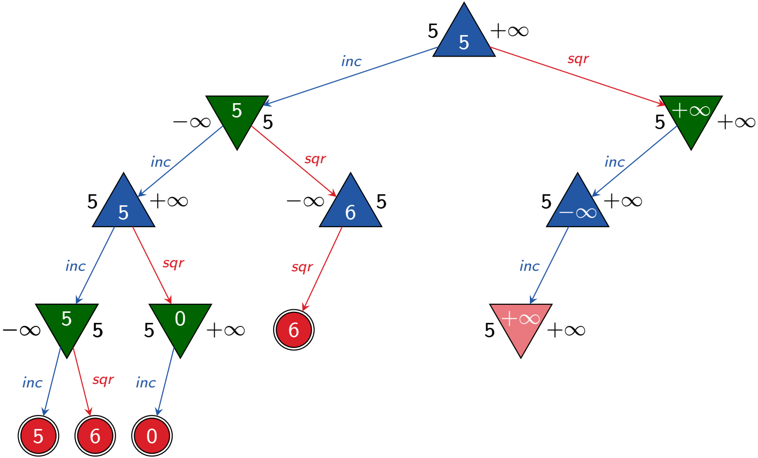

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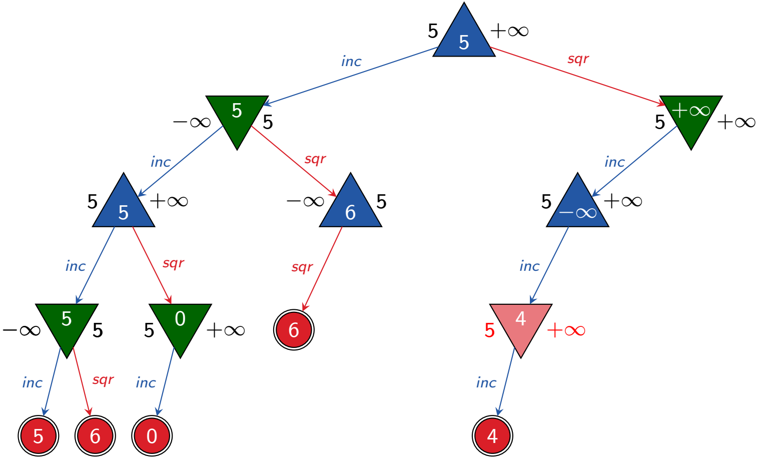

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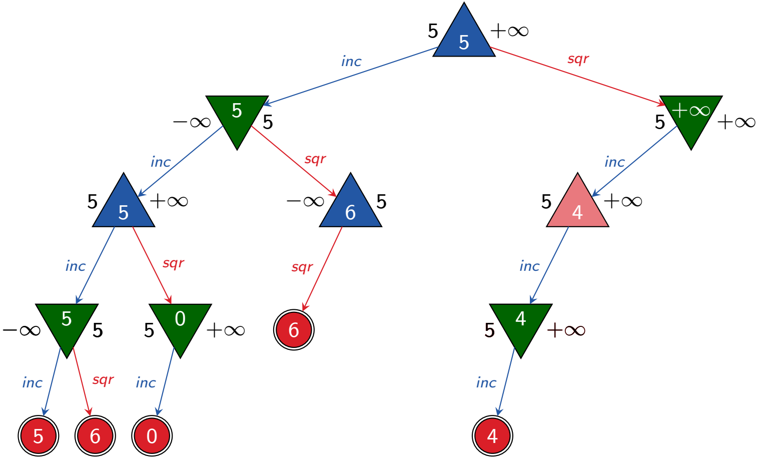

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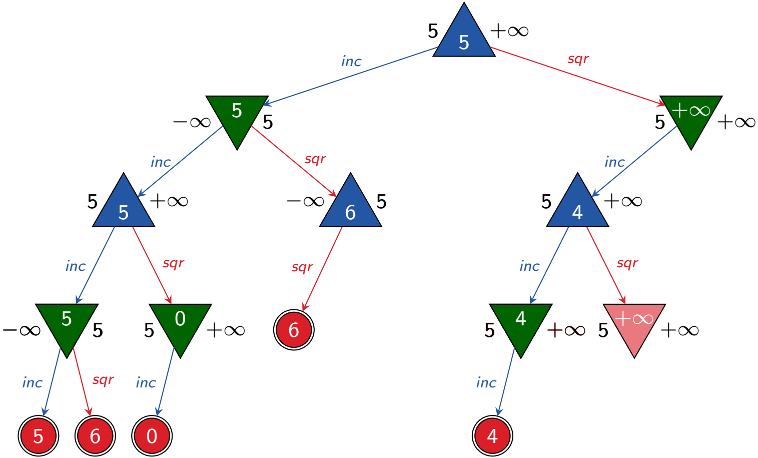

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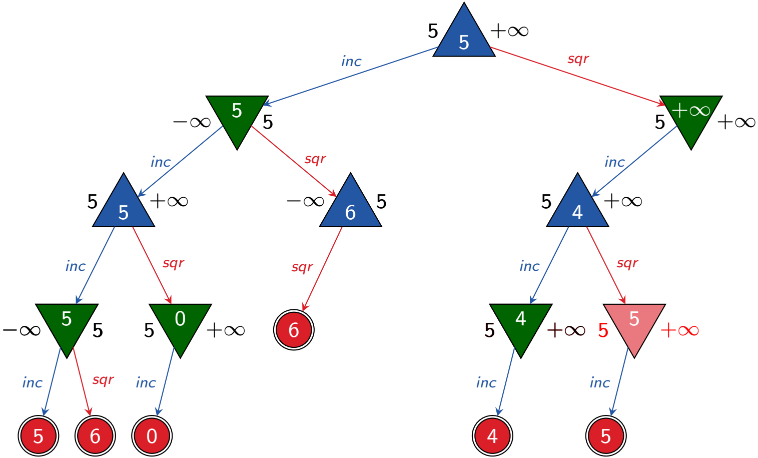

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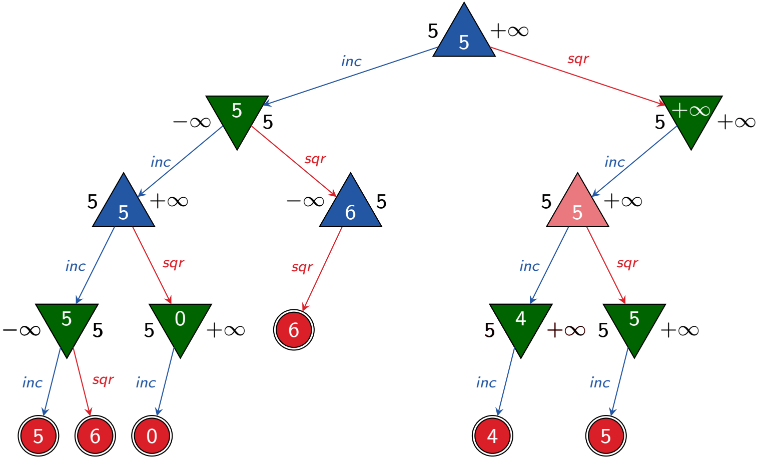

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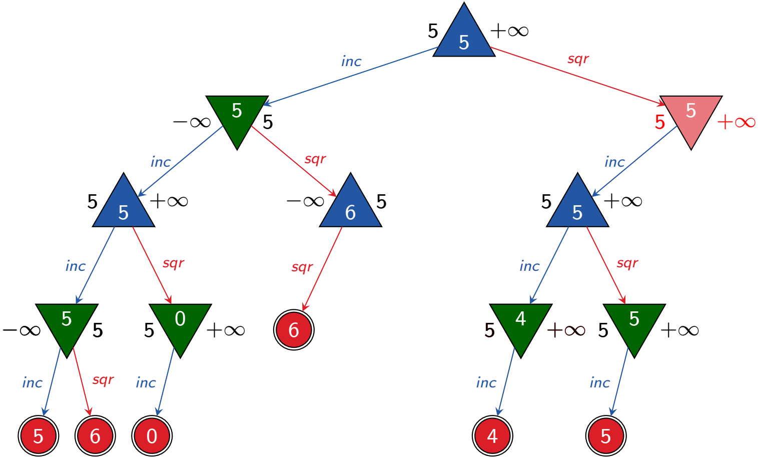

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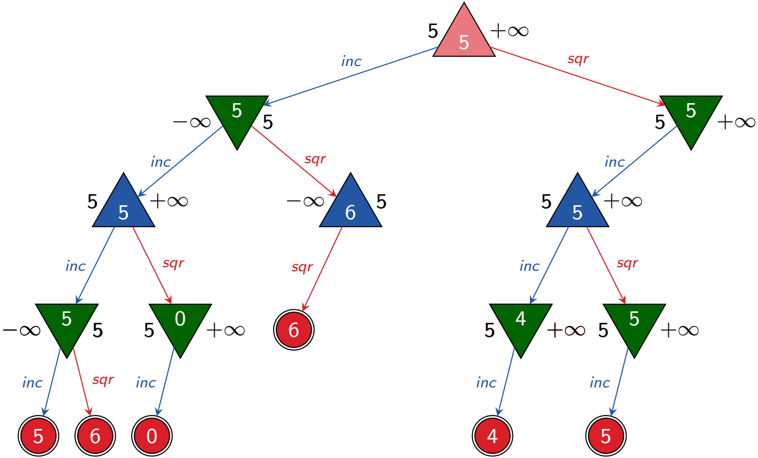

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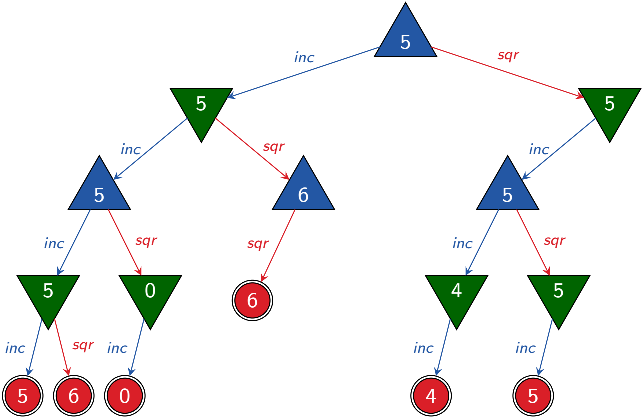

What do the utility values express?

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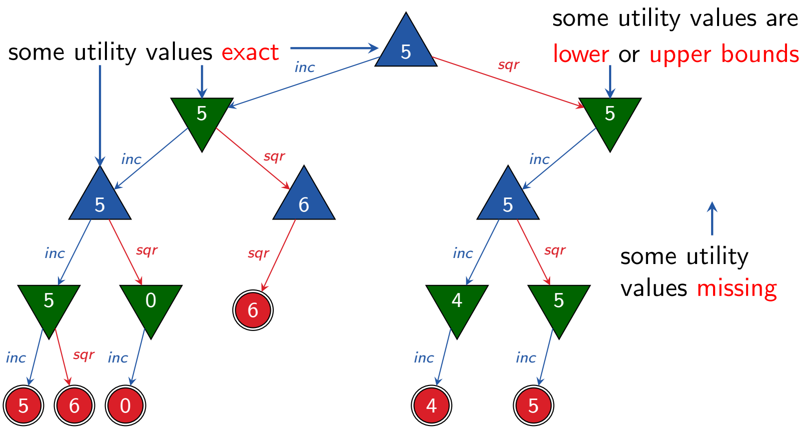
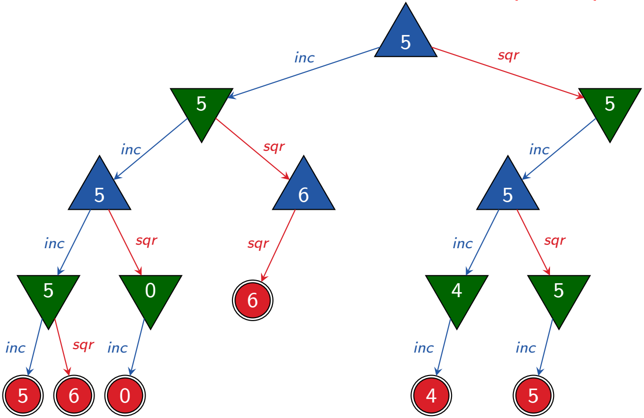


What do the utility values express?

Discussion

What does this mean for the computed policy?



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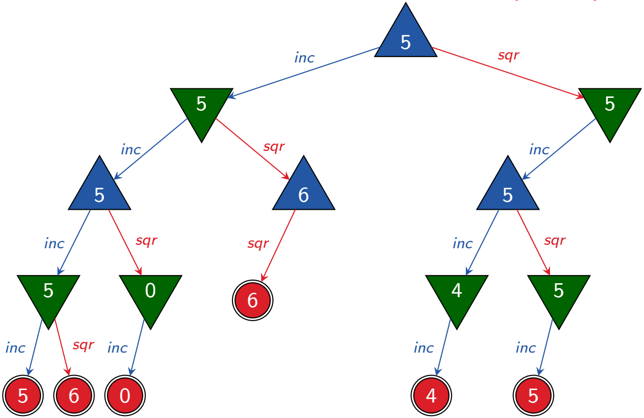
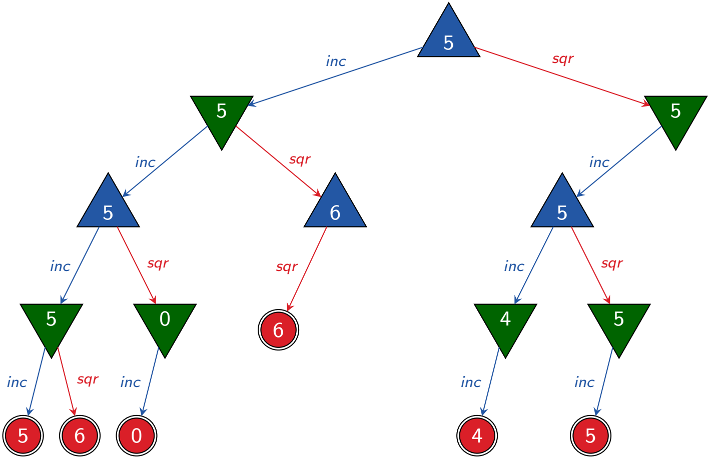


- only partial
- optimal in positions reachable under optimal play
- need to take earliest move in case of ties

Move Ordering

How Much Effort Do We Save?



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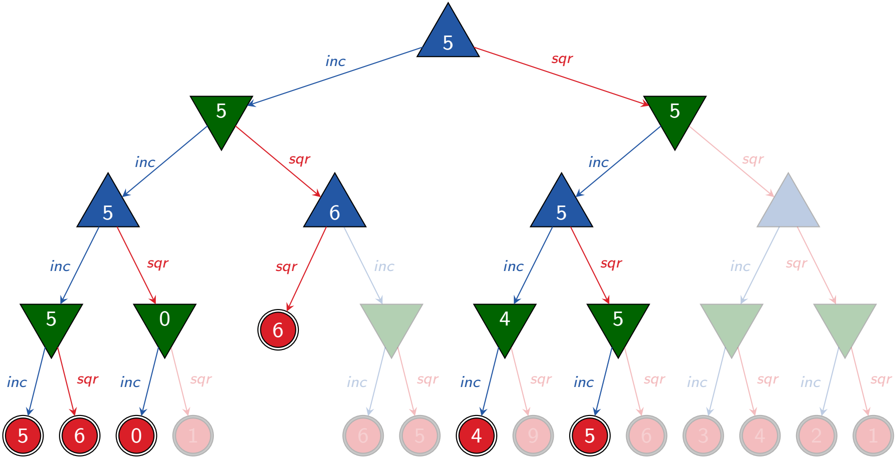
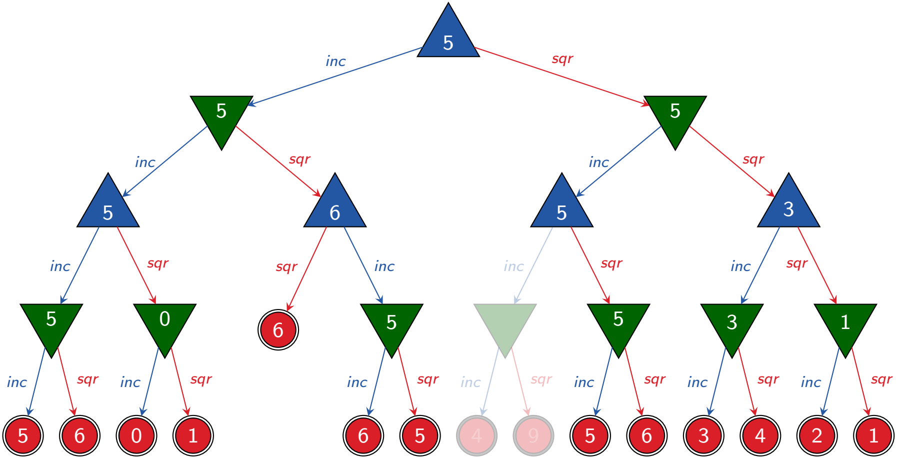


Were We Lucky?



if successors are considered in reverse order, we prune only a few positions

Move Ordering

idea: first consider the successors that are likely to be best

- domain-specific ordering function
  e.g., chess: captures $<$ threats $<$ forward moves $<$ backward moves
- dynamic move-ordering
  - first try moves that were good in the past
  - e.g., in iterative deepening search:
    best moves from previous iteration

How Much Do We Gain with Alpha-Beta Pruning?

assumption: uniform game tree, depth d, branching factor $b \geq 2$; MAX and MIN positions alternate

- perfect move ordering
  - best move at every position is considered first
  - maximizing move for MAX, minimizing move for MIN
  - effort reduced from $O(b^d)$ (minimax) to $O(b^{d/2})$
  - doubles the search depth that can be achieved in same time
- random move ordering
  - effort still reduced to $O(b^{3d/4})$

In practice, we can often get close to the perfect move ordering.

Heuristic Alpha-Beta Search

- combines evaluation function and alpha-beta search
- often uses additional techniques, e.g.
  - quiescence search
  - transposition tables
  - forward pruning
  - specialized subprocedures for certain parts of the game (e.g., opening libraries and endgame databases)
  - ...

Summary

Summary

alpha-beta search

- stores which utility both players can force somewhere else in the game tree
- exploits this information to avoid unnecessary computations
- can have significantly lower search effort than minimax
- best case: search twice as deep in the same time