Foundations of Artificial Intelligence

G3. Board Games: Alpha-Beta Search

Malte Helmert

University of Basel

May 19, 2025

Board Games: Overview

chapter overview:

- G1. Introduction and State of the Art
- G2. Minimax Search and Evaluation Functions
- G3. Alpha-Beta Search
- G4. Stochastic Games
- G5. Monte-Carlo Tree Search Framework
- G6. Monte-Carlo Tree Search Configurations

Limitations of Minimax

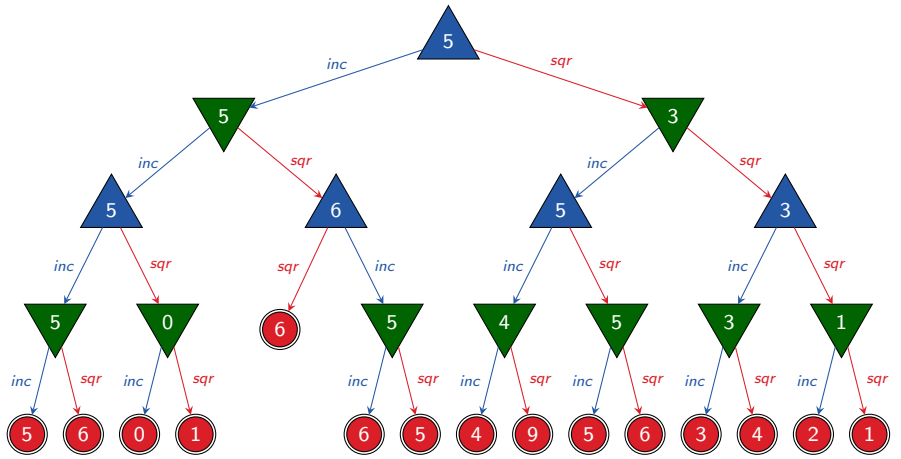
What if the size of the game tree is too big for minimax?

$\rightsquigarrow$ heuristic alpha-beta search

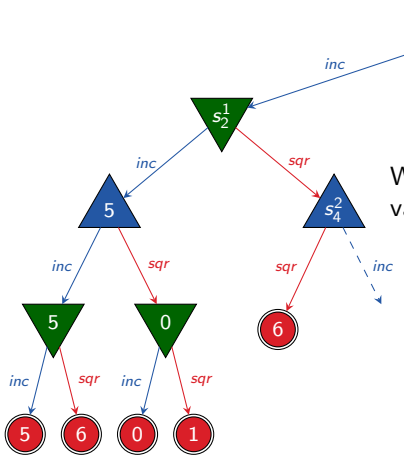
- heuristics (evaluation functions): previous chapter
- alpha-beta search: this chapter

G3.1 Alpha-Beta Search

Can We Save Search Effort?



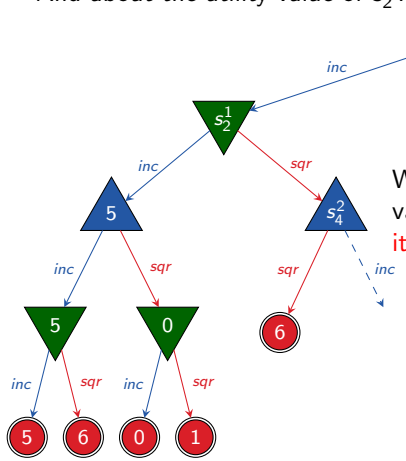
Can We Save Search Effort?



What do we know about the utility value of s_4^2 in this situation?

Can We Save Search Effort?

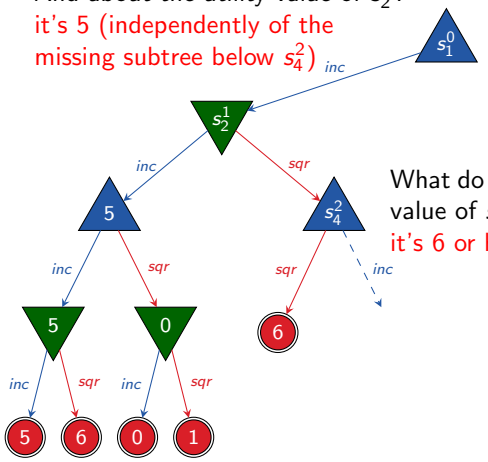
And about the utility value of s_2^1?



What do we know about the utility value of s_4^2 in this situation?
it's 6 or higher

Can We Save Search Effort?

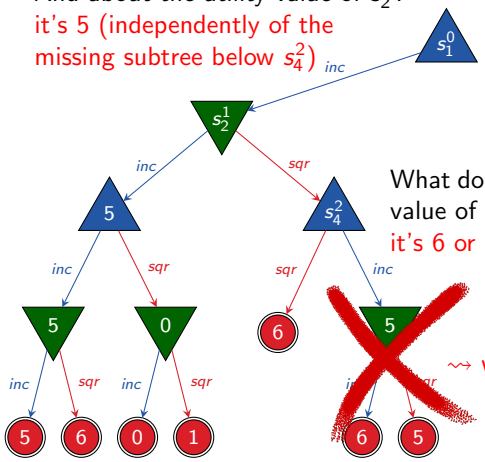
And about the utility value of s_2^1?
it's 5 (independently of the missing subtree below s_4^2)



What do we know about the utility value of s_4^2 in this situation?
it's 6 or higher

Can We Save Search Effort?

And about the utility value of s_2^1?
it's 5 (independently of the missing subtree below s_4^2)



What do we know about the utility value of s_4^2 in this situation?
it's 6 or higher

$\rightsquigarrow$ we don't have to look at this

Idea

idea: for every search node, use two values α and β such that we know that the subtree rooted at the node

- is irrelevant if its utility is $\leq \alpha$
  because MAX will prevent entering it when playing optimally
- is irrelevant if its utility is $\geq \beta$
  because MIN will prevent entering it when playing optimally

We can prune every node with $\alpha \geq \beta$
because it must be irrelevant (no matter what its utility is).

Alpha-Beta Search: Pseudo Code

- algorithm skeleton the same as minimax
- function signature extended by two variables α and β

function alpha-beta-main(p)
$\langle v, move \rangle :=$ alpha-beta($p, -\infty, +\infty$)
return *move*

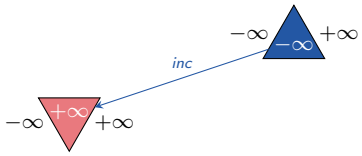
Alpha-Beta Search: Pseudo-Code

function alpha-beta(p, α, β)
if p is terminal position:
    return $\langle utility(p), \textbf{none} \rangle$
initialize v and *best_move* [as in minimax]
for each $\langle move, p' \rangle \in succ(p)$:
    $\langle v', best\_move' \rangle := alpha\text{-}beta(p', \alpha, \beta)$
    update v and *best_move* [as in minimax]
    if $player(p) = MAX$:
        if $v \geq \beta$:
            return $\langle v, \textbf{none} \rangle$
        $\alpha := \max\{\alpha, v\}$
    if $player(p) = MIN$:
        if $v \leq \alpha$:
            return $\langle v, \textbf{none} \rangle$
        $\beta := \min\{\beta, v\}$
return $\langle v, best\_move \rangle$

Example

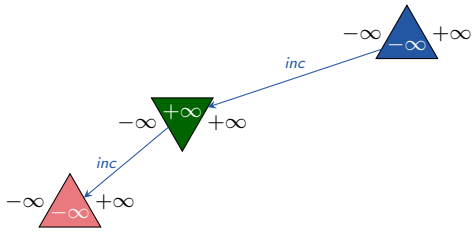
- α: lower bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- β: upper bound of relevant utility
- a MIN subtree is pruned if utility $\leq \alpha$

Example



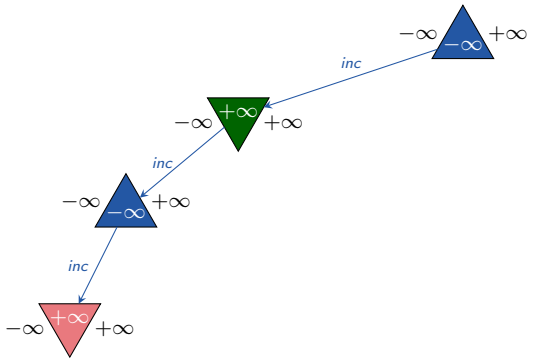
- α: lower bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- β: upper bound of relevant utility
- a MIN subtree is pruned if utility $\leq \alpha$

Example



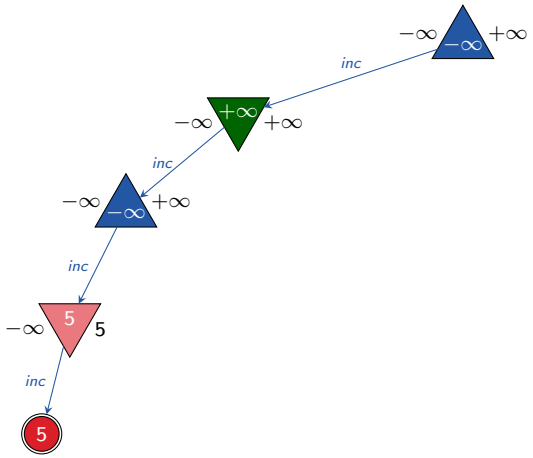
- α: lower bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- β: upper bound of relevant utility
- a MIN subtree is pruned if utility $\leq \alpha$

Example



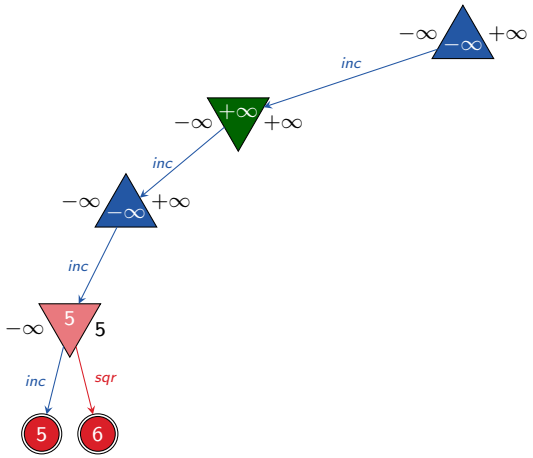
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



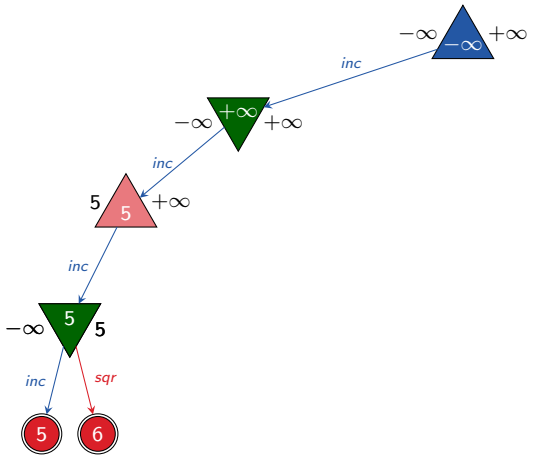
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



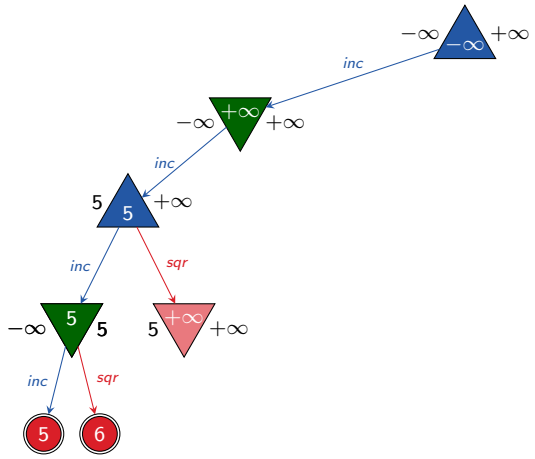
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



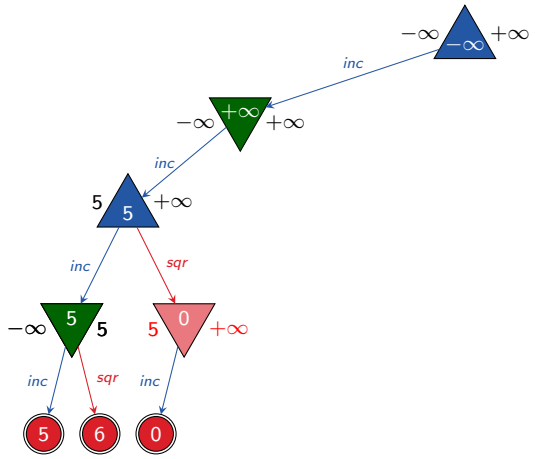
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



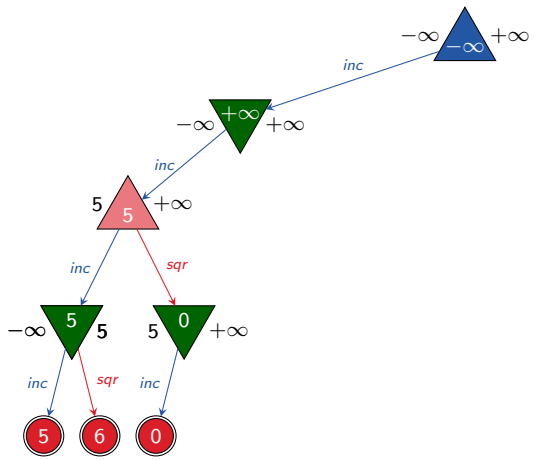
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



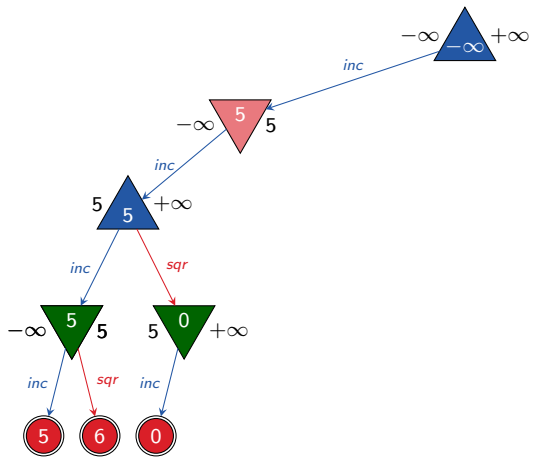
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



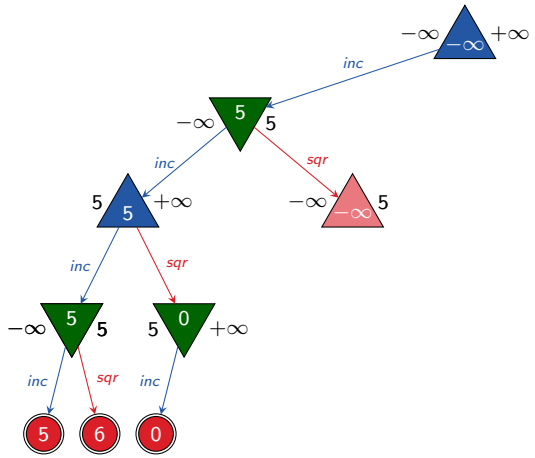
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



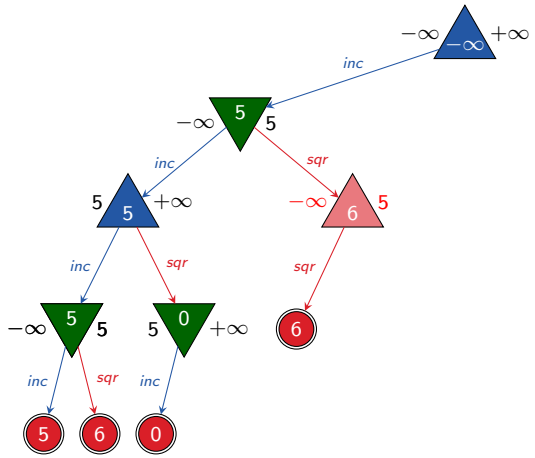
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



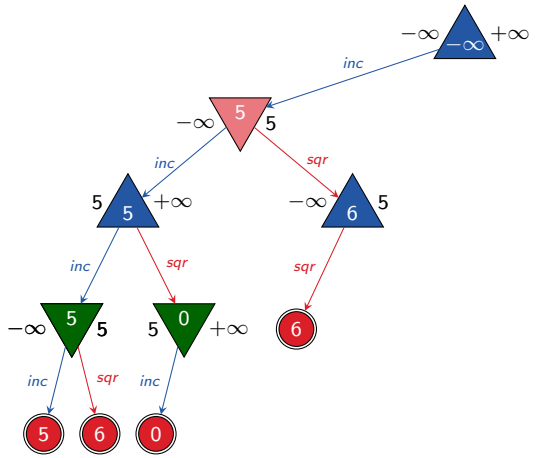
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



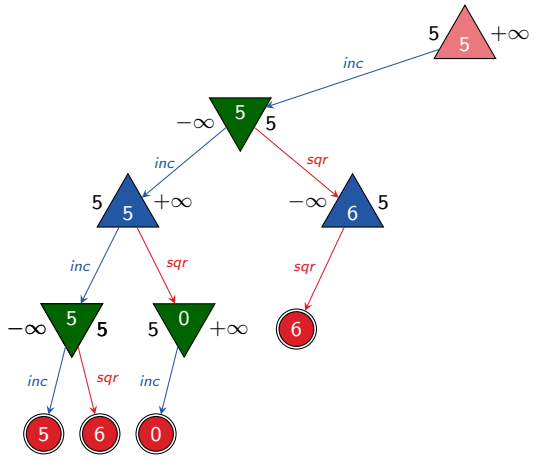
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



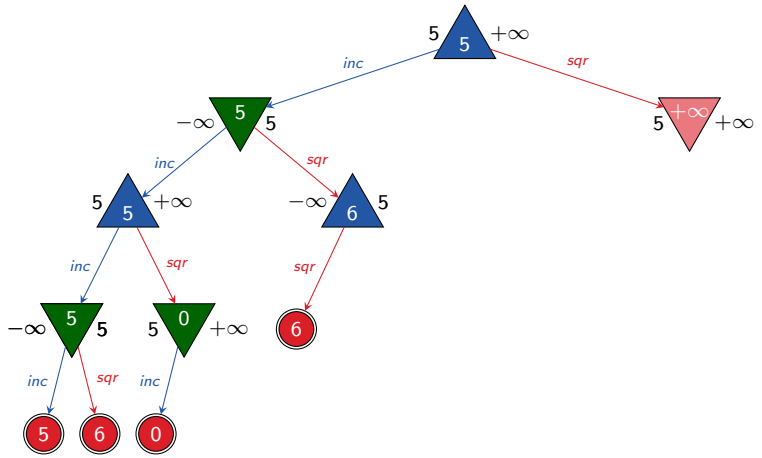
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



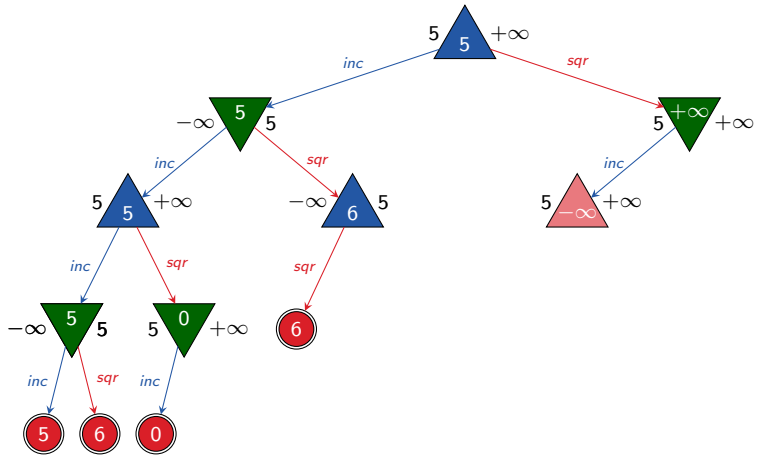
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



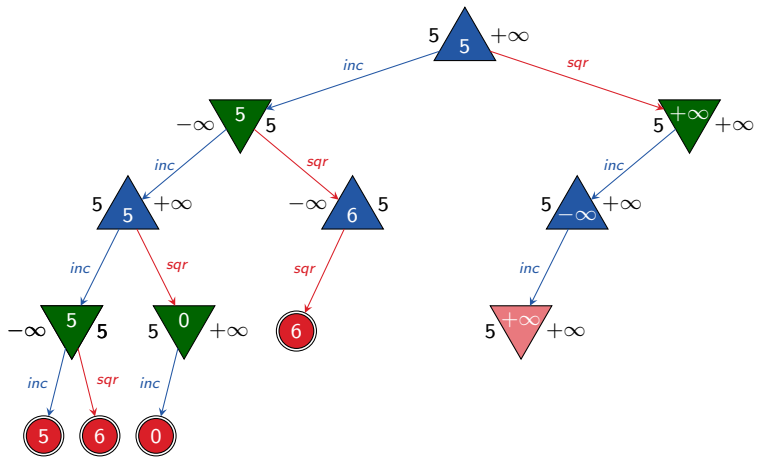
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



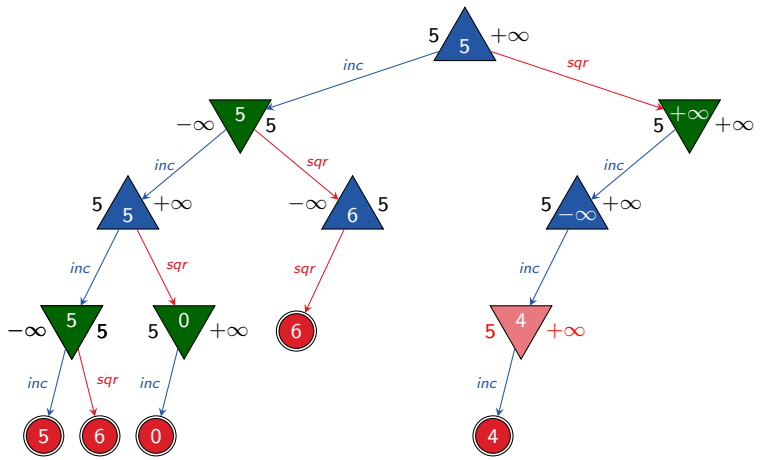
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



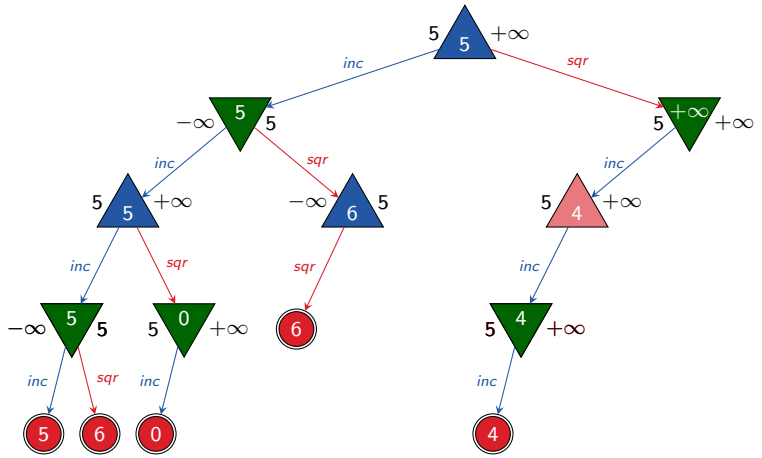
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



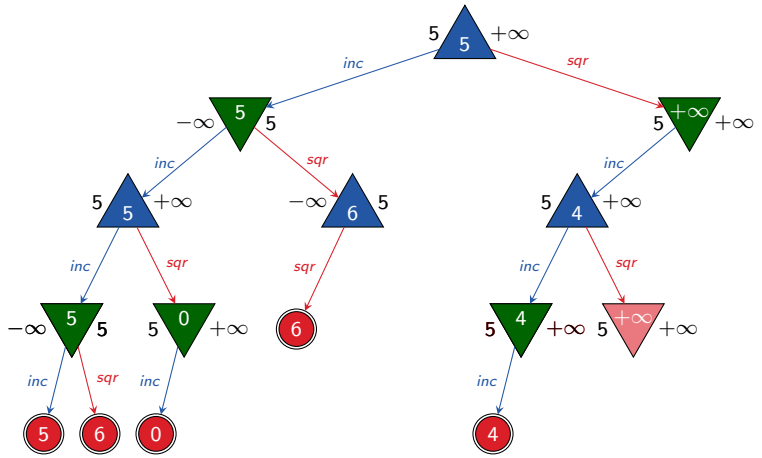
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



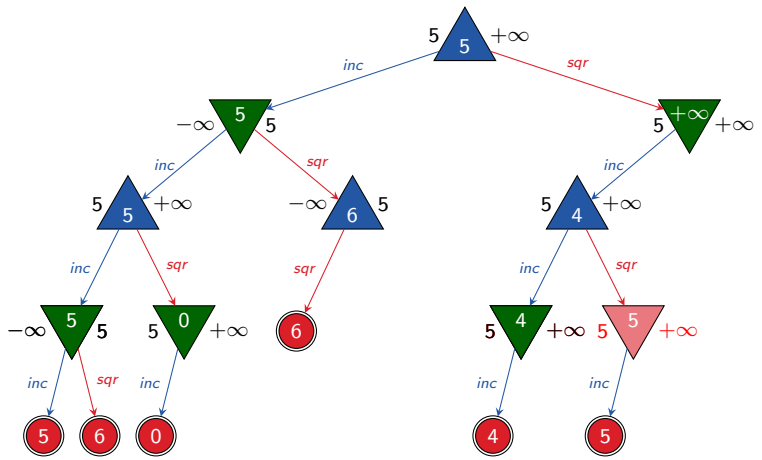
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



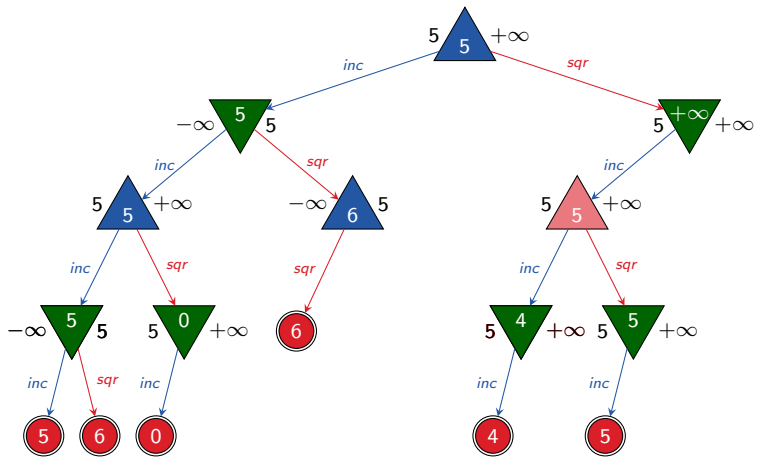
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



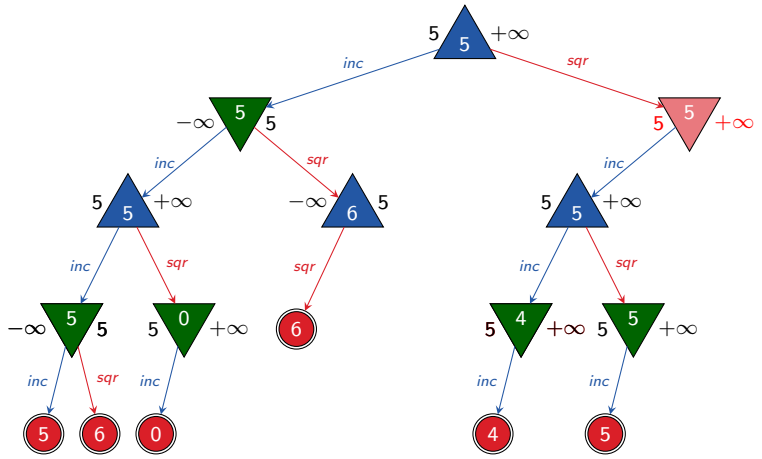
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



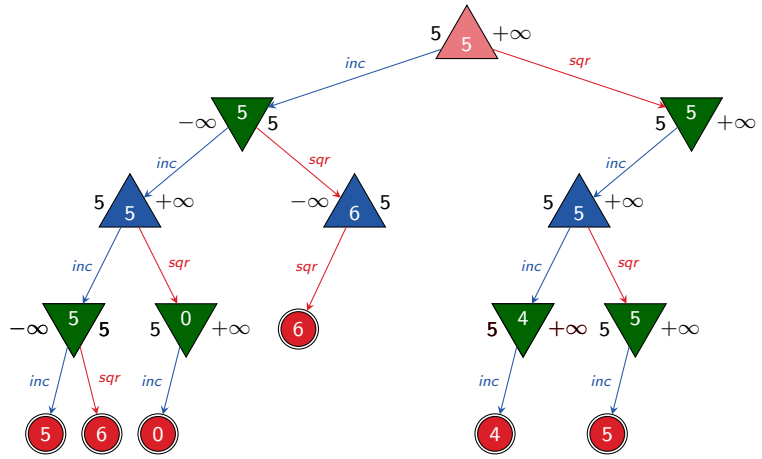
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



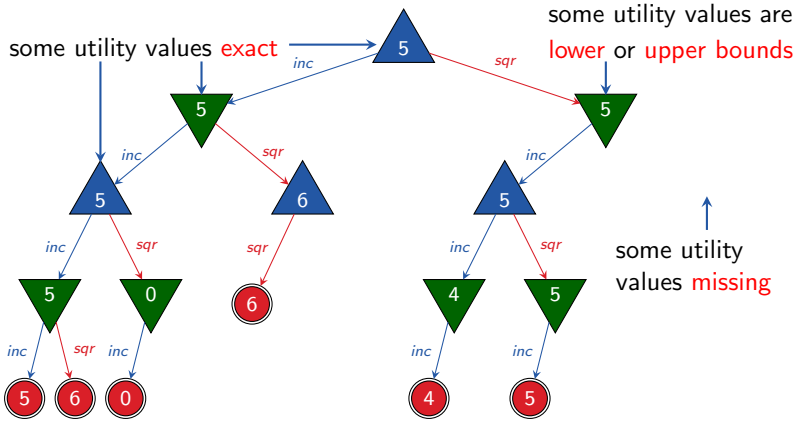
- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

Example



- α: lower bound of relevant utility
- β: upper bound of relevant utility
- a MAX subtree is pruned if utility $\geq \beta$
- a MIN subtree is pruned if utility $\leq \alpha$

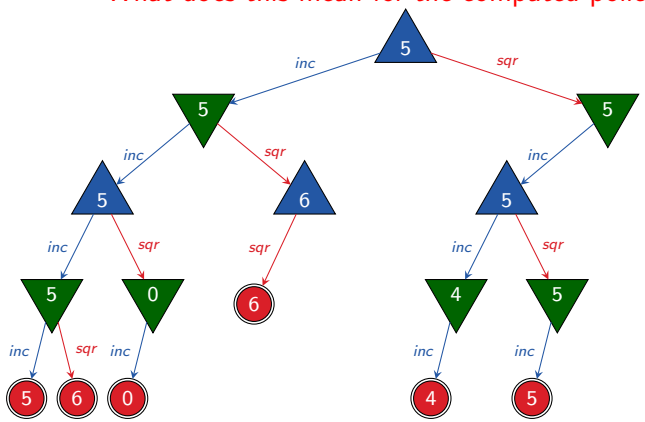
Discussion



What do the utility values express?

Discussion

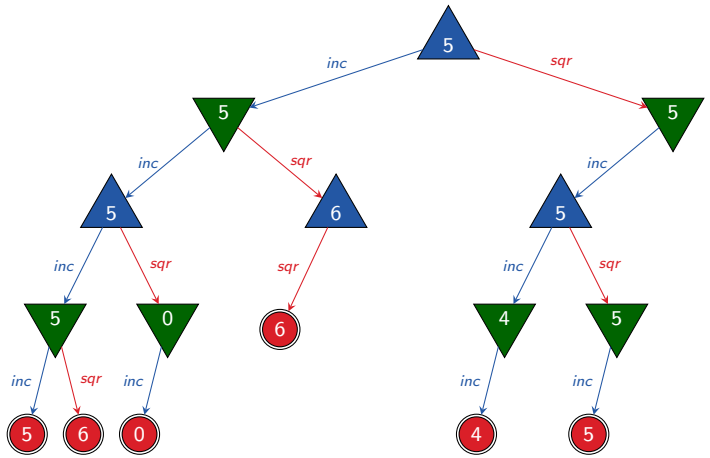
What does this mean for the computed policy?



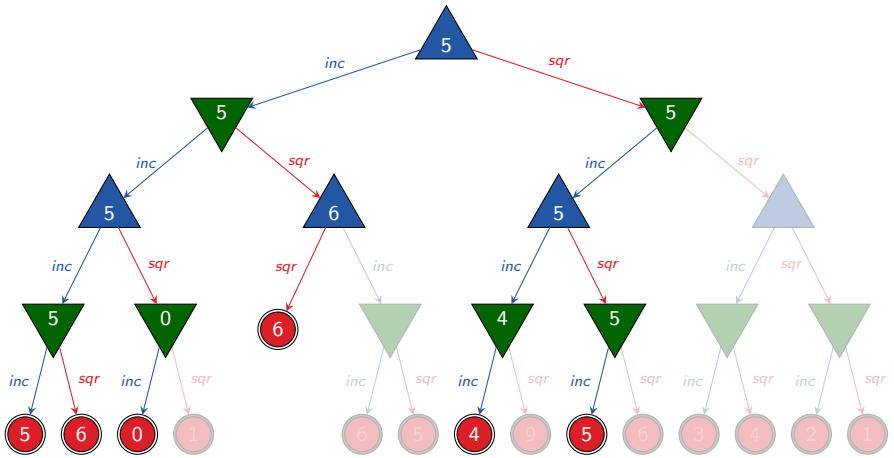
- only partial
- optimal in positions reachable under optimal play
- need to take earliest move in case of ties

G3.2 Move Ordering

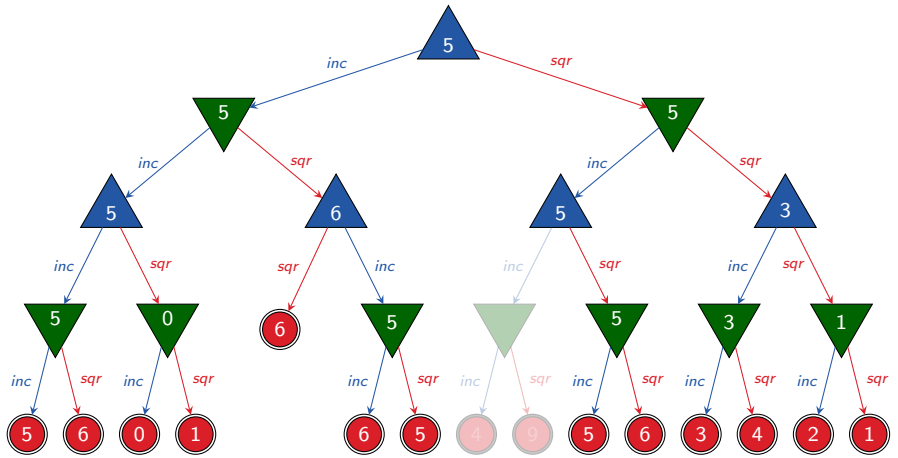
How Much Effort Do We Save?



How Much Effort Do We Save?



Were We Lucky?



if successors are considered in reverse order, we prune only a few positions

Move Ordering

idea: first consider the successors that are likely to be best

- domain-specific ordering function
  e.g., chess: captures $<$ threats $<$ forward moves $<$ backward moves
- dynamic move-ordering
  - first try moves that were good in the past
  - e.g., in iterative deepening search:
    best moves from previous iteration

How Much Do We Gain with Alpha-Beta Pruning?

assumption: uniform game tree, depth d, branching factor $b \geq 2$;
MAX and MIN positions alternate

- perfect move ordering
  - best move at every position is considered first
  - maximizing move for MAX, minimizing move for MIN
  - effort reduced from $O(b^d)$ (minimax) to $O(b^{d/2})$
  - doubles the search depth that can be achieved in same time
- random move ordering
  - effort still reduced to $O(b^{3d/4})$

In practice, we can often get close to the perfect move ordering.

Heuristic Alpha-Beta Search

- combines evaluation function and alpha-beta search
- often uses additional techniques, e.g.
  - quiescence search
  - transposition tables
  - forward pruning
  - specialized subprocedures for certain parts of the game
    (e.g., opening libraries and endgame databases)
  - ...

G3.3 Summary

Summary

alpha-beta search

- stores which utility both players can force somewhere else in the game tree
- exploits this information to avoid unnecessary computations
- can have significantly lower search effort than minimax
- best case: search twice as deep in the same time