

Foundations of Artificial Intelligence

E5. Propositional Logic: Local Search and Outlook

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence and Normal Forms
- E3. Reasoning and Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

Local Search: GSAT

Local Search for SAT

- Apart from systematic search, there are also successful **local search methods** for SAT.
- These are usually not complete and in particular cannot prove **unsatisfiability** for a formula.
- They are often still interesting because they can find models for hard problems.
- However, all in all, DPLL-based methods have been more successful in recent years.

Local Search for SAT: Ideas

local search methods directly applicable to SAT:

- **candidates:** (complete) assignments
- **solutions:** satisfying assignments
- **search neighborhood:** change assignment of **one** variable
- **heuristic:** depends on algorithm; e.g., #unsatisfied clauses

GSAT (Greedy SAT): Pseudo-Code

auxiliary functions:

- **violated**(Δ, I): number of clauses in Δ not satisfied by I
- **flip**(I, v): assignment that results from I when changing the valuation of proposition v

function GSAT(Δ):

repeat *max-tries* **times**:

$I :=$ a random assignment

repeat *max-flips* **times**:

if $I \models \Delta$:

return I

$V_{\text{greedy}} :=$ the set of variables v occurring in Δ
 for which **violated**($\Delta, \text{flip}(I, v)$) is minimal

 randomly select $v \in V_{\text{greedy}}$

$I := \text{flip}(I, v)$

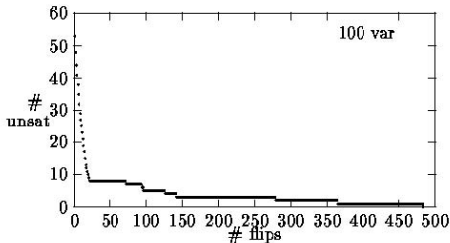
return no solution found

GSAT: Discussion

GSAT has the usual ingredients of local search methods:

- hill climbing
- randomness (although **relatively little!**)
- restarts

empirically, much time is spent on plateaus:



Local Search: Walksat

Walksat: Pseudo-Code

$\text{lost}(\Delta, I, v)$: #clauses in Δ satisfied by I , but not by $\text{flip}(I, v)$

function Walksat(Δ):

repeat *max-tries* **times**:

I := a random assignment

repeat *max-flips* **times**:

if $I \models \Delta$:

return I

C := randomly chosen unsatisfied clause in Δ

if there is a variable v in C with $\text{lost}(\Delta, I, v) = 0$:

V_{choices} := all such variables in C

else with probability p_{noise} :

V_{choices} := all variables occurring in C

else:

V_{choices} := variables v in C that minimize $\text{lost}(\Delta, I, v)$

 randomly select $v \in V_{\text{choices}}$

I := $\text{flip}(I, v)$

return no solution found

Walksat vs. GSAT

Comparison GSAT vs. Walksat:

- much more randomness in Walksat
because of random choice of considered clause
 - “counter-intuitive” steps that temporarily increase
the number of unsatisfied clauses are possible in Walksat
- ⇒ smaller risk of getting stuck in local minima

How Difficult Is SAT?

How Difficult is SAT in Practice?

- SAT is NP-complete.
- known algorithms like DPLL
need exponential time in the worst case
- What about the **average case**?
- depends on **how** the average is computed
(no “obvious” way to define the average)

SAT: Polynomial Average Runtime

Good News (Goldberg 1979)

construct random CNF formulas
with n variables and k clauses as follows:

In every clause, every variable occurs

- positively with probability $\frac{1}{3}$,
- negatively with probability $\frac{1}{3}$,
- not at all with probability $\frac{1}{3}$.

Then the runtime of DPLL in the average case
is polynomial in n and k .

↪ not a realistic model for practically relevant CNF formulas
(because almost all of the random formulas are satisfiable)

Phase Transitions

How to find **interesting** random problems?

conjecture of Cheeseman et al.:

Cheeseman et al., IJCAI 1991

Every NP-complete problem has at least one **size parameter** such that the difficult instances are close to a **critical value** of this parameter.

This so-called **phase transition** separates two problem regions, e.g., an **over-constrained** and an **under-constrained** region.

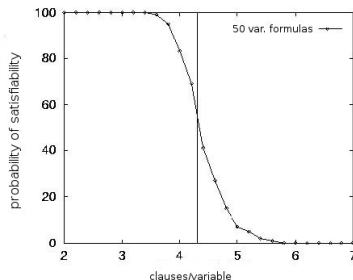
↪ confirmed for, e.g., graph coloring, Hamiltonian paths and **SAT**

Phase Transitions for 3-SAT

Problem Model of Mitchell et al., AAAI 1992

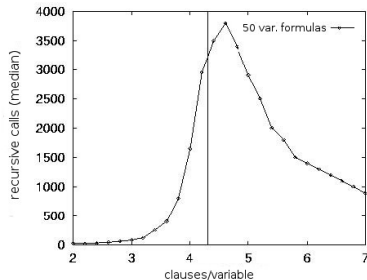
- fixed clause size of 3
- in every clause, choose the variables randomly
- literals positive or negative with equal probability

critical parameter: $\# \text{clauses} \div \# \text{variables}$
phase transition at ratio ≈ 4.3



Phase Transition of DPLL

DPLL shows high runtime close to the phase transition region:



Phase Transition: Intuitive Explanation

- If there are **many** clauses and hence the instance is unsatisfiable with high probability, this can be shown efficiently with unit propagation.
- If there are **few** clauses, there are many satisfying assignments, and it is easy to find one of them.
- Close to the **phase transition**, there are many “almost-solutions” that have to be considered by the search algorithm.

Outlook

State of the Art

- research on SAT in general:
 ↪ <http://www.satlive.org/>
- conferences on SAT since 1996 (annually since 2000)
 ↪ <http://www.satisfiability.org/>
- competitions for SAT algorithms since 1992
 ↪ <http://www.satcompetition.org/>
 - largest instances have more than 1 000 000 literals
 - different tracks (e.g., SAT vs. SAT+UNSAT;
 industrial vs. random instances)

More Advanced Topics

DPLL-based SAT algorithms:

- efficient implementation techniques
- accurate variable orders
- clause learning

local search algorithms:

- efficient implementation techniques
- adaptive search methods (“difficult” clauses are recognized after some time and then prioritized)

SAT modulo theories:

- extension with background theories (e.g., real numbers, data structures, ...)

Summary

Summary (1)

- **local search** for SAT searches in the space of interpretations; neighbors: assignments that differ only in one variable
- has typical properties of local search methods: evaluation functions, randomization, restarts
- example: **GSAT** (Greedy SAT)
 - hill climbing with heuristic function: #unsatisfied clauses
 - randomization through tie-breaking and restarts
- example: **Walksat**
 - focuses on **randomly selected** unsatisfied clauses
 - does not follow the heuristic always, but also **injects noise**
 - consequence: **more randomization** as GSAT and lower risk of getting stuck in local minima

Summary (2)

- **more detailed analysis** of SAT shows: the problem is NP-complete, but not all instances are difficult
- randomly generated SAT instances are easy to satisfy if they contain few clauses, and easy to prove unsatisfiable if they contain many clauses
- in between: **phase transition**