Foundations of Artificial Intelligence

E5. Propositional Logic: Local Search and Outlook

Malte Helmert

University of Basel

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence and Normal Forms
- E3. Reasoning and Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

E5.1 Local Search: GSAT

Local Search for SAT

- Apart from systematic search, there are also successful local search methods for SAT.
- These are usually not complete and in particular cannot prove unsatisfiability for a formula.
- They are often still interesting because they can find models for hard problems.
- However, all in all, DPLL-based methods have been more successful in recent years.

Local Search for SAT: Ideas

local search methods directly applicable to SAT:

- candidates: (complete) assignments
- solutions: satisfying assignments
- search neighborhood: change assignment of one variable
- heuristic: depends on algorithm; e.g., #unsatisfied clauses

GSAT (Greedy SAT): Pseudo-Code

auxiliary functions:

- violated(Δ, I): number of clauses in Δ not satisfied by I
- flip(I, v): assignment that results from I when changing the valuation of proposition v

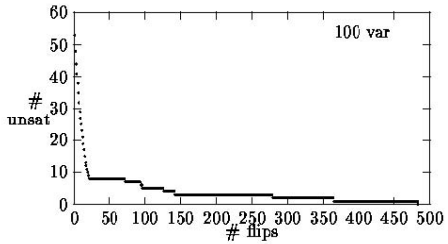
function GSAT(Δ):
repeat *max-tries* **times**:
  $I :=$ a random assignment
  repeat *max-flips* **times**:
    if $I \models \Delta$:
      return I
    $V_{\text{greedy}} :=$ the set of variables v occurring in Δ
        for which violated$(\Delta, \text{flip}(I, v))$ is minimal
    randomly select $v \in V_{\text{greedy}}$
    $I := \text{flip}(I, v)$
return no solution found

GSAT: Discussion

GSAT has the usual ingredients of local search methods:

- hill climbing
- randomness (although relatively little!)
- restarts

empirically, much time is spent on plateaus:



E5.2 Local Search: Walksat

Walksat: Pseudo-Code

lost(Δ, I, v): #clauses in Δ satisfied by I, but not by flip(I, v)

function Walksat(Δ):
repeat *max-tries* **times**:
  $I :=$ a random assignment
  repeat *max-flips* **times**:
    if $I \models \Delta$:
      return I
    $C :=$ randomly chosen unsatisfied clause in Δ
    if there is a variable v in C with lost$(\Delta, I, v) = 0$:
      $V_{\text{choices}} :=$ all such variables in C
    else with probability p_{noise}:
      $V_{\text{choices}} :=$ all variables occurring in C
    else:
      $V_{\text{choices}} :=$ variables v in C that minimize lost(Δ, I, v)
    randomly select $v \in V_{\text{choices}}$
    $I :=$ flip(I, v)
return no solution found

Walksat vs. GSAT

Comparison GSAT vs. Walksat:

- much more randomness in Walksat because of random choice of considered clause
- "counter-intuitive" steps that temporarily increase the number of unsatisfied clauses are possible in Walksat

$\rightsquigarrow$ smaller risk of getting stuck in local minima

E5.3 How Difficult Is SAT?

How Difficult is SAT in Practice?

- SAT is NP-complete.

⇝ known algorithms like DPLL
need exponential time in the worst case

- What about the average case?
- depends on how the average is computed
(no "obvious" way to define the average)

SAT: Polynomial Average Runtime

Good News (Goldberg 1979)

construct random CNF formulas
with n variables and k clauses as follows:

In every clause, every variable occurs

- positively with probability $\frac{1}{3}$,
- negatively with probability $\frac{1}{3}$,
- not at all with probability $\frac{1}{3}$.

Then the runtime of DPLL in the average case
is polynomial in n and k.

⇝ not a realistic model for practically relevant CNF formulas
(because almost all of the random formulas are satisfiable)

Phase Transitions

How to find interesting random problems?

conjecture of Cheeseman et al.:

Cheeseman et al., IJCAI 1991

Every NP-complete problem has at least one size parameter
such that the difficult instances are close to a critical value
of this parameter.

This so-called phase transition separates two problem regions,
e.g., an over-constrained and an under-constrained region.

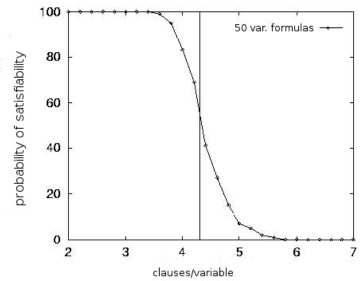
⇝ confirmed for, e.g., graph coloring, Hamiltonian paths and SAT

Phase Transitions for 3-SAT

Problem Model of Mitchell et al., AAAI 1992

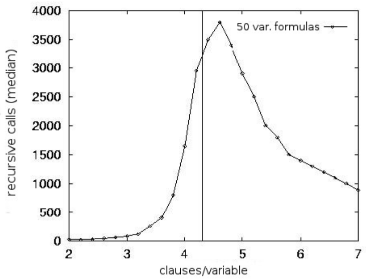
- fixed clause size of 3
- in every clause, choose the variables randomly
- literals positive or negative with equal probability

critical parameter: #clauses divided by #variables
phase transition at ratio ≈ 4.3



Phase Transition of DPLL

DPLL shows high runtime close to the phase transition region:



Phase Transition: Intuitive Explanation

- If there are many clauses and hence the instance is unsatisfiable with high probability, this can be shown efficiently with unit propagation.
- If there are few clauses, there are many satisfying assignments, and it is easy to find one of them.
- Close to the phase transition, there are many "almost-solutions" that have to be considered by the search algorithm.

E5.4 Outlook

State of the Art

- research on SAT in general:
  $\leadsto$ `http://www.satlive.org/`
- conferences on SAT since 1996 (annually since 2000)
  $\leadsto$ `http://www.satisfiability.org/`
- competitions for SAT algorithms since 1992
  $\leadsto$ `http://www.satcompetition.org/`
  - largest instances have more than 1 000 000 literals
  - different tracks (e.g., SAT vs. SAT+UNSAT; industrial vs. random instances)

More Advanced Topics

DPLL-based SAT algorithms:

- efficient implementation techniques
- accurate variable orders
- clause learning

local search algorithms:

- efficient implementation techniques
- adaptive search methods ("difficult" clauses are recognized after some time and then prioritized)

SAT modulo theories:

- extension with background theories (e.g., real numbers, data structures, ...)

E5.5 Summary

Summary (1)

- local search for SAT searches in the space of interpretations; neighbors: assignments that differ only in one variable
- has typical properties of local search methods: evaluation functions, randomization, restarts
- example: GSAT (Greedy SAT)
  - hill climbing with heuristic function: #unsatisfied clauses
  - randomization through tie-breaking and restarts
- example: Walksat
  - focuses on randomly selected unsatisfied clauses
  - does not follow the heuristic always, but also injects noise
  - consequence: more randomization as GSAT and lower risk of getting stuck in local minima

Summary (2)

- more detailed analysis of SAT shows: the problem is NP-complete, but not all instances are difficult
- randomly generated SAT instances are easy to satisfy if they contain few clauses, and easy to prove unsatisfiable if they contain many clauses
- in between: phase transition