Foundations of Artificial Intelligence

E2. Propositional Logic: Equivalence and Normal Forms

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence and Normal Forms
- E3. Reasoning and Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

E2.1 Equivalence

Logical Equivalance

Definition (logically equivalent)

Formulas φ and ψ are called logically equivalent ($\varphi \equiv \psi$) if for all interpretations I: $I \models \varphi$ iff $I \models \psi$.

German: logisch äquivalent

Equivalences

Logical Equivalences

Let φ, ψ, and η be formulas.

- $(\varphi \land \psi) \equiv (\psi \land \varphi)$ and $(\varphi \lor \psi) \equiv (\psi \lor \varphi)$ (commutativity)
- $((\varphi \land \psi) \land \eta) \equiv (\varphi \land (\psi \land \eta))$ and $((\varphi \lor \psi) \lor \eta) \equiv (\varphi \lor (\psi \lor \eta))$ (associativity)
- $((\varphi \land \psi) \lor \eta) \equiv ((\varphi \lor \eta) \land (\psi \lor \eta))$ and $((\varphi \lor \psi) \land \eta) \equiv ((\varphi \land \eta) \lor (\psi \land \eta))$ (distributivity)
- $\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$ and $\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi)$ (De Morgan)
- $\neg\neg\varphi \equiv \varphi$ (double negation)
- $(\varphi \to \psi) \equiv (\neg\varphi \lor \psi)$ (($\to$)-elimination)

Commutativity and associativity are often used implicitly
$\rightsquigarrow$ We write $(X_1 \land X_2 \land X_3 \land X_4)$ instead of $(X_1 \land (X_2 \land (X_3 \land X_4)))$

E2.2 Normal Forms

Normal Forms: Terminology

Definition (literal)

If $P \in \Sigma$, then the formulas P and $\neg P$ are called literals.

P is called positive literal, $\neg P$ is called negative literal.

The complementary literal to P is $\neg P$ and vice versa.
For a literal ℓ, the complementary literal to ℓ is denoted with $\bar{\ell}$.

German: Literal, positives/negatives/komplementäres Literal

Question: What is the difference between $\bar{\ell}$ and $\neg\ell$?

Normal Forms: Terminology

Definition (clause)

A disjunction of 0 or more literals is called a clause.
The empty clause (with 0 literals) is $\bot$.
Clauses consisting of exactly one literal are called unit clauses.

German: Klausel, leere Klausel, Einheitsklausel

Definition (monomial)

A conjunction of 0 or more literals is called a monomial.

German: Monom

Normal Forms

Definition (normal forms)

A formula φ is in conjunctive normal form (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula φ is in disjunctive normal form (DNF) if φ is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

German: konjunktive Normalform, disjunktive Normalform

Normal Forms

For every propositional formula, there exists a logically equivalent propositional formula in CNF and in DNF.

Conversion to CNF with equivalences

1. eliminate implications
   $(\varphi \rightarrow \psi) \equiv (\neg\varphi \lor \psi)$ (($\rightarrow$)-elimination)
2. move negations inside
   $\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$ (De Morgan)
   $\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi)$ (De Morgan)
   $\neg\neg\varphi \equiv \varphi$ (double negation)
3. distribute $\lor$ over $\land$
   $((\varphi \land \psi) \lor \eta) \equiv ((\varphi \lor \eta) \land (\psi \lor \eta))$ (distributivity)
4. simplify constant subformulas ($\top, \bot$)

There are formulas φ for which every logically equivalent formula in CNF and DNF is exponentially longer than φ.

E2.3 Summary

Summary

- two formulas are logically equivalent if they have the same models
- different kinds of formulas:
  - atomic formulas and literals
  - clauses and monomials
  - conjunctive normal form (CNF) and disjunctive normal form (DNF)
- for every formula, there is a logically equivalent formula in CNF and a logically equivalent formula in DNF