

Foundations of Artificial Intelligence

E2. Propositional Logic: Equivalence and Normal Forms

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E2.1 Equivalence

E2.2 Normal Forms

E2.3 Summary

Propositional Logic: Overview

Chapter overview: propositional logic

- ▶ E1. Syntax and Semantics
- ▶ E2. Equivalence and Normal Forms
- ▶ E3. Reasoning and Resolution
- ▶ E4. DPLL Algorithm
- ▶ E5. Local Search and Outlook

E2.1 Equivalence

Logical Equivalence

Definition (logically equivalent)

Formulas φ and ψ are called **logically equivalent** ($\varphi \equiv \psi$) if for all interpretations I : $I \models \varphi$ iff $I \models \psi$.

German: logisch äquivalent

Equivalences

Logical Equivalences

Let φ , ψ , and η be formulas.

- ▶ $(\varphi \wedge \psi) \equiv (\psi \wedge \varphi)$ and $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$ (commutativity)
- ▶ $((\varphi \wedge \psi) \wedge \eta) \equiv (\varphi \wedge (\psi \wedge \eta))$ and
 $((\varphi \vee \psi) \vee \eta) \equiv (\varphi \vee (\psi \vee \eta))$ (associativity)
- ▶ $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$ and
 $((\varphi \vee \psi) \wedge \eta) \equiv ((\varphi \wedge \eta) \vee (\psi \wedge \eta))$ (distributivity)
- ▶ $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ and
 $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$ (De Morgan)
- ▶ $\neg\neg\varphi \equiv \varphi$ (double negation)
- ▶ $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ ((\rightarrow) -elimination)

Commutativity and associativity are often used implicitly

\rightsquigarrow We write $(X_1 \wedge X_2 \wedge X_3 \wedge X_4)$ instead of $(X_1 \wedge (X_2 \wedge (X_3 \wedge X_4)))$

E2.2 Normal Forms

Normal Forms: Terminology

Definition (literal)

If $P \in \Sigma$, then the formulas P and $\neg P$ are called **literals**.

P is called **positive literal**, $\neg P$ is called **negative literal**.

The **complementary literal** to P is $\neg P$ and vice versa.

For a literal ℓ , the complementary literal to ℓ is denoted with $\bar{\ell}$.

German: Literal, positives/negatives/komplementäres Literal

Question: What is the difference between $\bar{\ell}$ and $\neg\ell$?

Normal Forms: Terminology

Definition (clause)

A disjunction of 0 or more literals is called a **clause**.

The **empty clause** (with 0 literals) is \perp .

Clauses consisting of exactly one literal are called **unit clauses**.

German: Klausel, leere Klausel, Einheitsklausel

Definition (monomial)

A conjunction of 0 or more literals is called a **monomial**.

German: Monom

Normal Forms

Definition (normal forms)

A formula φ is in **conjunctive normal form** (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^n \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula φ is in **disjunctive normal form** (DNF) if φ is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^n \left(\bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

German: konjunktive Normalform, disjunktive Normalform

Normal Forms

For every propositional formula, there exists a logically equivalent propositional formula in CNF and in DNF.

Conversion to CNF with equivalences

- ① eliminate implications

$$(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi) \quad ((\rightarrow)\text{-elimination})$$

- ② move negations inside

$$\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi) \quad (\text{De Morgan})$$

$$\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi) \quad (\text{De Morgan})$$

$$\neg\neg\varphi \equiv \varphi \quad (\text{double negation})$$

- ③ distribute \vee over \wedge

$$((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta)) \quad (\text{distributivity})$$

- ④ simplify constant subformulas (\top, \perp)

There are formulas φ for which every logically equivalent formula in CNF and DNF is exponentially longer than φ .

E2.3 Summary

Summary

- ▶ two formulas are **logically equivalent** if they have the same models
- ▶ different kinds of formulas:
 - ▶ **atomic formulas** and **literals**
 - ▶ **clauses** and **monomials**
 - ▶ **conjunctive normal form** (CNF) and **disjunctive normal form** (DNF)
- ▶ for every formula, there is a logically equivalent formula in CNF and a logically equivalent formula in DNF