Foundations of Artificial Intelligence E1. Propositional Logic: Syntax and Semantics

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence and Normal Forms
- E3. Reasoning and Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

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Classification

classification:

Propositional Logic

environment:

- static vs. dynamic
- deterministic vs. non-deterministic vs. stochastic
- fully vs. partially vs. not observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

• problem-specific vs. general vs. learning

(applications also in more complex environments)

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Motivation

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Propositional Logic: Motivation

propositional logic

- modeling and representing problems and knowledge
- basis for general problem descriptions and solving strategies
 → automated planning (Part F)
- allows for automated reasoning

German: Aussagenlogik, automatisches Schliessen

Relationship to CSPs

- previous part: constraint satisfaction problems
- satisfiability problem in propositional logic can be viewed as non-binary CSP over {F, T}
- formula encodes constraints
- solution: satisfying assignment of values to variables
- backtracking with inference \approx DPLL (Chapter E4)

Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

```
two-missionaries-are-on-left-shore
one-cannibal-is-on-left-shore
boat-is-on-left-shore
```

. . .

• problem description for general problem solvers

• states represented as truth values of atomic propositions German: Aussagenvariablen

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Propositional Logic: Intuition

propositions: atomic statements over the world that cannot be divided further

Propositions with logical connectives like "and", "or" and "not" form the propositional formulas.

German: logische Verknüpfungen

Syntax and Semantics

Today, we define syntax and semantics of propositional logic. ~> reminder from Discrete Mathematics in Computer Science

syntax:

- defines which symbols are allowed in formulas
 (,), ℵ, ∧, A, B, C, X, ♡, →, ↗, ...?
- ... and which sequences of these symbols are correct formulas $(A \land B)$, $((A) \land B)$, $\land)A(B, ...?$

semantics:

- defines the meaning of formulas
- uses interpretations to describe a possible world $I = \{A \mapsto \mathbf{T}, B \mapsto \mathbf{F}\}$
- tells us which formulas are true in which worlds

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Alphabet of Propositions

Syntax

- Logical formulas use an alphabet Σ of propositions, for example Σ = {P, Q, R, S} or Σ = {X₁, X₂, X₃,...}.
- We do not mention the alphabet in the following.
- More formally, all definitions are parameterized by Σ .

German: Alphabet

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Syntax

Definition (propositional formula)

- \top and \perp are formulas (constant true/constant false).
- Every proposition in Σ is a formula (atomic formula).
- If φ is a formula, then $\neg \varphi$ is a formula (negation).
- If φ and ψ are formulas, then so are
 - $(\varphi \land \psi)$ (conjunction)
 - $(\varphi \lor \psi)$ (disjunction)
 - $(\varphi \rightarrow \psi)$ (implication)

German: aussagenlogische Formel, konstant wahr/falsch, atomare Formel, Konjunktion, Disjunktion, Implikation

Note: minor differences to Discrete Mathematics course

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may omit redundant parentheses:

- outer parentheses of formula:
 - $(P \land Q) \lor R$ instead of $((P \land Q) \lor R)$

Abbreviating Notations: Omitting Parenthesis

- multiple conjunctions/disjunctions:
 - $P \land Q \land \neg R \land S$ instead of $(((P \land Q) \land \neg R) \land S)$
- implicit binding strength: $(\neg) > (\land) > (\lor) > (\rightarrow)$:
 - $P \lor Q \land R$ instead of $P \lor (Q \land R)$
 - use responsibly

Abbreviating Notations: Prefix Notation

prefix notations used like \sum for sums:

•
$$\bigvee_{i=1}^{4} X_i$$
 instead of $(X_1 \lor X_2 \lor X_3 \lor X_4)$
• $\bigwedge_{i=1}^{3} Y_i$ instead of $(Y_1 \land Y_2 \land Y_3)$

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Semantics

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Intuition for Semantics

A formula is true or false depending on the interpretation of the propositions.

Semantics: Intuition

- A proposition *P* is either true or false. The truth value of *P* is determined by an interpretation.
- The truth value of a formula follows from the truth values of the propositions.

Example

example interpretations for $\varphi = (P \lor Q) \land R$:

- If P and Q are false and R is true, then φ is false.
- If P is false and Q and R are true, then φ is true.

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Interpre	tations			

Definition (interpretation)

An interpretation I is a function $I : \Sigma \to {\mathbf{T}, \mathbf{F}}.$

Interpretations are sometimes called truth assignments.

German: Interpretation/Belegung/Wahrheitsbelegung

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The Semantics of Formulas

When is a formula φ true under interpretation *I*? symbolically: When does $I \models \varphi$ hold?

Definition (Models and the \models Relation)

The relation " \models " is a relation between interpretations and formulas and is defined as follows:

•
$$I \models \top$$
 and $I \not\models \bot$
• $I \models P$ if $I(P) = \mathbf{T}$ for $P \in \Sigma$
• $I \models \neg \varphi$ if $I \not\models \varphi$
• $I \models (\varphi \land \psi)$ if $I \models \varphi$ and $I \models \psi$
• $I \models (\varphi \lor \psi)$ if $I \models \varphi$ or $I \models \psi$
• $I \models (\varphi \rightarrow \psi)$ if $I \not\models \varphi$ or $I \models \psi$
If $I \models \varphi$ ($I \not\models \varphi$), we say φ is true (false) under I .

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Examples

Example (Interpretation *I*)

$$I = \{P \mapsto \mathsf{T}, Q \mapsto \mathsf{T}, R \mapsto \mathsf{F}, S \mapsto \mathsf{F}\}$$

Which formulas are true under *I*?

•
$$\varphi_1 = \neg (P \land Q) \land (R \land \neg S)$$
. Does $I \models \varphi_1$ hold?

•
$$\varphi_2 = (P \land Q) \land \neg (R \land \neg S)$$
. Does $I \models \varphi_2$ hold?

•
$$\varphi_3 = (R \rightarrow P)$$
. Does $I \models \varphi_3$ hold?

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Models of Formulas and Sets of Formulas

Definition (model)

An interpretation I is called a model of φ if $I \models \varphi$.

Definition $(I \models \Phi)$

Let Φ be a set of propositional formulas. We write $I \models \Phi$ if $I \models \varphi$ for all $\varphi \in \Phi$. Such an interpretation I is called a model of Φ .

If I is a model of formula φ , we also say "I satisfies φ " or " φ holds under I" (similarly for sets of formulas Φ).

German: Modell, erfüllt, gilt unter

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Satisfiable, Unsatisfiable, Falsifiable, Valid

Definition (satisfiable etc.)

A formula φ is called

- \bullet satisfiable if there exists a model for φ
- unsatisfiable if there exists no model for φ
- valid (= a tautology) if all interpretations are models of φ
- $\bullet~{\rm falsifiable}$ if not all interpretations are models of φ

German: erfüllbar, unerfüllbar, allgemeingültig (gültig, Tautologie), falsifizierbar

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Truth Tables

How to determine automatically if a given formula is (un)satisfiable, valid, falsifiable?

 \rightsquigarrow simple method: truth tables

example: Is $\varphi = ((P \lor H) \land \neg H) \to P$ valid?

Р	Н	$P \lor H$	$((P \lor H) \land \neg H)$	$((P \lor H) \land \neg H) \to P$
F	F	F	F	Т
F	Т	Т	F	Т
Т	F	Т	Т	Т
Т	Т	Т	F	Т

 $I \models \varphi$ for all interpretations I, hence φ is valid.

• Is it satisfiable/unsatisfiable/falsifiable?

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Properties of Formulas

Terminology (Side Note)

What does " φ is true" mean?

- not formally defined
- the statement misses an interpretation
 - could be meant as "in the obvious interpretation" in some cases
 - or as "in all possible interpretations" (tautology)
- imprecise language → avoid

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Summary



- Propositional logic forms the basis for a general representation of problems and knowledge.
- Propositions (atomic formulas) are statements over the world that cannot be divided further.
- Propositional formulas combine constant and atomic formulas with ¬, ∧, ∨, → to more complex statements.
- Interpretations determine which atomic formulas are true and which ones are false.
- Interpretations making a formula true are called models.
- important properties a formula may have: satisfiable, unsatisfiable, valid, falsifiable