Foundations of Artificial Intelligence

E1. Propositional Logic: Syntax and Semantics

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence and Normal Forms
- E3. Reasoning and Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

Classification

classification:

Propositional Logic

environment:

- static vs. dynamic
- deterministic vs. non-deterministic vs. stochastic
- fully vs. partially vs. not observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. general vs. learning

(applications also in more complex environments)

E1.1 Motivation

Propositional Logic: Motivation

propositional logic

- modeling and representing problems and knowledge
- basis for general problem descriptions and solving strategies
  $\rightsquigarrow$ automated planning (Part F)
- allows for automated reasoning

German: Aussagenlogik, automatisches Schliessen

Relationship to CSPs

- previous part: constraint satisfaction problems
- satisfiability problem in propositional logic can be viewed as non-binary CSP over $\{\mathbf{F}, \mathbf{T}\}$
- formula encodes constraints
- solution: satisfying assignment of values to variables
- backtracking with inference $\approx$ DPLL (Chapter E4)

Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

```
two-missionaries-are-on-left-shore
one-cannibal-is-on-left-shore
boat-is-on-left-shore
...
```

- problem description for general problem solvers
- states represented as truth values of atomic propositions

German: Aussagenvariablen

Propositional Logic: Intuition

propositions: atomic statements over the world
that cannot be divided further

Propositions with logical connectives like
"and", "or" and "not" form the propositional formulas.

German: logische Verknüpfungen

Syntax and Semantics

Today, we define syntax and semantics of propositional logic.
$\rightsquigarrow$ reminder from Discrete Mathematics in Computer Science

syntax:

- defines which symbols are allowed in formulas
  $(,), \aleph, \wedge, A, B, C, X, \heartsuit, \rightarrow, \nearrow, \dots$?
- ... and which sequences of these symbols are correct formulas
  $(A \wedge B), \quad ((A) \wedge B), \quad \wedge)A(B, \dots$?

semantics:

- defines the meaning of formulas
- uses interpretations to describe a possible world
  $I = \{A \mapsto \mathbf{T}, B \mapsto \mathbf{F}\}$
- tells us which formulas are true in which worlds

E1.2 Syntax

Alphabet of Propositions

- Logical formulas use an alphabet Σ of propositions, for example $\Sigma = \{P, Q, R, S\}$ or $\Sigma = \{X_1, X_2, X_3, \dots\}$.
- We do not mention the alphabet in the following.
- More formally, all definitions are parameterized by Σ.

German: Alphabet

Syntax

Definition (propositional formula)

- $\top$ and $\bot$ are formulas (constant true/constant false).
- Every proposition in Σ is a formula (atomic formula).
- If φ is a formula, then $\neg\varphi$ is a formula (negation).
- If φ and ψ are formulas, then so are
  - $(\varphi \land \psi)$ (conjunction)
  - $(\varphi \lor \psi)$ (disjunction)
  - $(\varphi \rightarrow \psi)$ (implication)

German: aussagenlogische Formel, konstant wahr/falsch, atomare Formel, Konjunktion, Disjunktion, Implikation

Note: minor differences to Discrete Mathematics course

Abbreviating Notations: Omitting Parenthesis

may omit redundant parentheses:

- outer parentheses of formula:
  - $(P \wedge Q) \vee R$ instead of $((P \wedge Q) \vee R)$
- multiple conjunctions/disjunctions:
  - $P \wedge Q \wedge \neg R \wedge S$ instead of $(((P \wedge Q) \wedge \neg R) \wedge S)$
- implicit binding strength: $(\neg) > (\wedge) > (\vee) > (\rightarrow)$:
  - $P \vee Q \wedge R$ instead of $P \vee (Q \wedge R)$
  - use responsibly

Abbreviating Notations: Prefix Notation

prefix notations used like $\sum$ for sums:

- $\bigvee_{i=1}^{4} X_i$ instead of $(X_1 \lor X_2 \lor X_3 \lor X_4)$
- $\bigwedge_{i=1}^{3} Y_i$ instead of $(Y_1 \land Y_2 \land Y_3)$

E1.3 Semantics

Intuition for Semantics

A formula is true or false
depending on the interpretation of the propositions.

Semantics: Intuition

- A proposition P is either true or false.
  The truth value of P is determined by an interpretation.
- The truth value of a formula follows from
  the truth values of the propositions.

Example

example interpretations for $\varphi = (P \vee Q) \wedge R$:

- If P and Q are false and R is true, then φ is false.
- If P is false and Q and R are true, then φ is true.

Interpretations

Definition (interpretation)

An interpretation I is a function $I : \Sigma \to \{\mathbf{T}, \mathbf{F}\}$.

Interpretations are sometimes called truth assignments.

German: Interpretation/Belegung/Wahrheitsbelegung

The Semantics of Formulas

When is a formula φ true under interpretation I?
symbolically: When does $I \models \varphi$ hold?

Definition (Models and the $\models$ Relation)

The relation "$\models$" is a relation between interpretations and formulas and is defined as follows:

- $I \models \top$ and $I \not\models \bot$
- $I \models P$ if $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \models \neg\varphi$ if $I \not\models \varphi$
- $I \models (\varphi \land \psi)$ if $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \lor \psi)$ if $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ if $I \not\models \varphi$ or $I \models \psi$

If $I \models \varphi$ ($I \not\models \varphi$), we say φ is true (false) under I.

Examples

Example (Interpretation I)

$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$

Which formulas are true under I?

- $\varphi_1 = \neg(P \land Q) \land (R \land \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \land Q) \land \neg(R \land \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

E1.4 Properties of Formulas

Models of Formulas and Sets of Formulas

Definition (model)

An interpretation I is called a model of φ if $I \models \varphi$.

Definition ($I \models \Phi$)

Let Φ be a set of propositional formulas.
We write $I \models \Phi$ if $I \models \varphi$ for all $\varphi \in \Phi$.
Such an interpretation I is called a model of Φ.

If I is a model of formula φ, we also say "I satisfies φ" or "φ holds under I" (similarly for sets of formulas Φ).

German: Modell, erfüllt, gilt unter

Satisfiable, Unsatisfiable, Falsifiable, Valid

Definition (satisfiable etc.)

A formula φ is called

- satisfiable if there exists a model for φ
- unsatisfiable if there exists no model for φ
- valid ($=$ a tautology) if all interpretations are models of φ
- falsifiable if not all interpretations are models of φ

German: erfüllbar, unerfüllbar, allgemeingültig (gültig, Tautologie), falsifizierbar

Truth Tables

Truth Tables

How to determine automatically if a given formula is (un)satisfiable, valid, falsifiable?

$\rightsquigarrow$ simple method: truth tables

example: Is $\varphi = ((P \lor H) \land \neg H) \to P$ valid?

P	H	$P \lor H$	$((P \lor H) \land \neg H)$	$((P \lor H) \land \neg H) \to P$
F	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**
T	**T**	**T**	**F**	**T**

$I \models \varphi$ for all interpretations I, hence φ is valid.

- Is it satisfiable/unsatisfiable/falsifiable?

Terminology (Side Note)

What does "φ is true" mean?

- not formally defined
- the statement misses an interpretation
  - could be meant as "in the obvious interpretation" in some cases
  - or as "in all possible interpretations" (tautology)
- imprecise language $\rightsquigarrow$ avoid

E1.5 Summary

Summary

- Propositional logic forms the basis for a general representation of problems and knowledge.
- Propositions (atomic formulas) are statements over the world that cannot be divided further.
- Propositional formulas combine constant and atomic formulas with $\neg$, $\wedge$, $\vee$, $\rightarrow$ to more complex statements.
- Interpretations determine which atomic formulas are true and which ones are false.
- Interpretations making a formula true are called models.
- important properties a formula may have: satisfiable, unsatisfiable, valid, falsifiable