Foundations of Artificial Intelligence

D7. Constraint Satisfaction Problems: Decomposition Methods

Malte Helmert

University of Basel

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
- D6–D7. Problem Structure
  - D6. Constraint Graphs
  - D7. Decomposition Methods

D7.1 Decomposition Methods

More Complex Graphs

What if the constraint graph is not a tree
and does not decompose into several components?

- idea 1: conditioning
- idea 2: tree decomposition

German: Konditionierung, Baumzerlegung

D7.2 Conditioning

Conditioning

Conditioning

idea: Apply backtracking with forward checking until the constraint graph restricted to the remaining unassigned variables decomposes or is a tree.

remaining problem ⇝ algorithms for simple constraint graphs

cutset conditioning:
Choose variable order such that early variables form a small cutset (i.e., set of variables such that removing these variables results in an acyclic constraint graph).

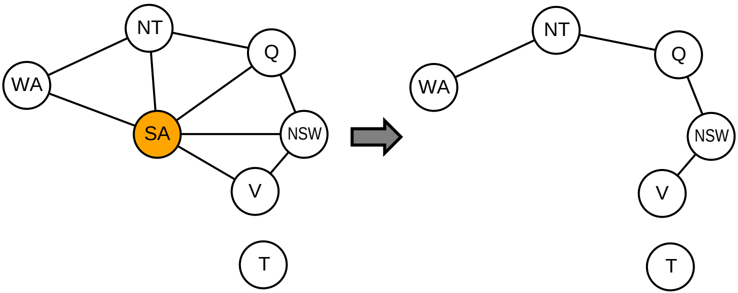
German: Cutset

time complexity: n variables, $m < n$ in cutset,
maximal domain size k: $O(k^m \cdot (n - m)k^2)$

(Finding optimal cutsets is an NP-complete problem.)

Conditioning: Example

Australia example: Cutset of size 1 suffices:



D7.3 Tree Decomposition

Tree Decomposition

basic idea of tree decomposition:

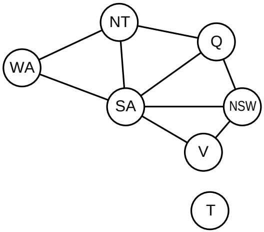
- Decompose constraint network into smaller subproblems (overlapping).
- Find solutions for the subproblems.
- Build overall solution based on the subsolutions.

more details:

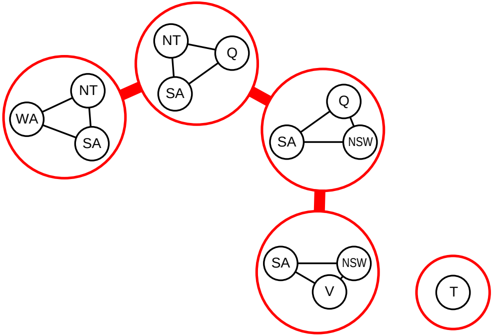
- "Overall solution building problem" based on subsolutions is a constraint network itself (meta constraint network).
- Choose subproblems in a way that the constraint graph of the meta constraint network is a tree/forest.
  $\rightsquigarrow$ build overall solution with efficient tree algorithm

Tree Decomposition: Example

constraint network:



tree decomposition:



Tree Decomposition: Definition

Definition (tree decomposition)

Consider a constraint network $\mathcal{C}$ with variables V.

A tree decomposition of $\mathcal{C}$
is a graph $\mathcal{T}$ with the following properties.

requirements on vertices:

- Every vertex of $\mathcal{T}$ corresponds to a subset of the variables V. Such a vertex (and corresponding variable set) is called a subproblem of $\mathcal{C}$.
- Every variable of V appears in at least one subproblem of $\mathcal{T}$.
- For every nontrivial constraint R_{uv} of $\mathcal{C}$, the variables u and v appear together in at least one subproblem in $\mathcal{T}$.

...

Tree Decomposition: Definition

Definition (tree decomposition)

Consider a constraint network $\mathcal{C}$ with variables V.

A tree decomposition of $\mathcal{C}$
is a graph $\mathcal{T}$ with the following properties.

...

requirements on edges:

- For each variable $v \in V$, let $\mathcal{T}_v$ be the set of vertices corresponding to the subproblems that contain v.
- For each variable v, the set $\mathcal{T}_v$ is connected, i.e., each vertex in $\mathcal{T}_v$ is reachable from every other vertex in $\mathcal{T}_v$ without visiting vertices not contained in $\mathcal{T}_v$.
- $\mathcal{T}$ is acyclic (a tree/forest)

Meta Constraint Network

meta constraint network $\mathcal{C}^{\mathcal{T}} = \langle V^{\mathcal{T}}, \mathrm{dom}^{\mathcal{T}}, (R^{\mathcal{T}}_{uv})\rangle$
based on tree decomposition $\mathcal{T}$

- $V^{\mathcal{T}} :=$ vertices of $\mathcal{T}$ (i.e., subproblems of $\mathcal{C}$ occurring in $\mathcal{T}$)
- $\mathrm{dom}^{\mathcal{T}}(v) :=$ set of solutions of subproblem v
- $R^{\mathcal{T}}_{uv} := \{\langle s, t\rangle \mid s, t$ compatible solutions of subproblems $u, v\}$ if $\{u, v\}$ is an edge of $\mathcal{T}$. (All constraints between subproblems not connected by an edge of $\mathcal{T}$ are trivial.)

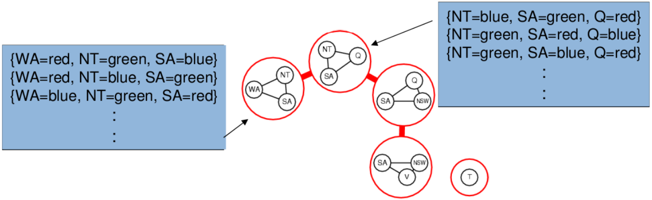
German: Meta-Constraintnetz

Solutions of two subproblems are called compatible
if all overlapping variables are assigned identically.

Solving with Tree Decompositions: Algorithm

algorithm:

- Find all solutions for all subproblems in the decomposition and build a tree-like meta constraint network.
- Constraints in meta constraint network: subsolutions must be compatible.
- Solve meta constraint network with an algorithm for tree-like networks.



Good Tree Decompositions

goal: each subproblem has as few variables as possible

- crucial: subproblem V' in $\mathcal{T}$ with highest number of variables
- number of variables in V' minus 1
  is called width of the decomposition
- best width over all decompositions: tree width
  of the constraint graph (computation is NP-complete)

time complexity of solving algorithm based on tree decompositions:
$O(nk^{w+1})$, where w is width of decomposition
(requires specialized version of revise; otherwise $O(nk^{2w+2})$.)

D7.4 Summary

Summary: This Chapter

- Reduce complex constraint graphs to simple constraint graphs.
- cutset conditioning:
  - Choose as few variables as possible (cutset) such that an assignment to these variables yields a remaining problem which is structurally simple.
  - search over assignments of variables in cutset
- tree decomposition: build tree-like meta constraint network
  - meta variables: groups of original variables that jointly cover all variables and constraints
  - values correspond to consistent assignments to the groups
  - constraints between overlapping groups to ensure compatibility
  - overall algorithm exponential in width of decomposition (size of largest group)

Summary: CSPs

Constraint Satisfaction Problems (CSP)

General formalism for problems where

- values have to be assigned to variables
- such that the given constraints are satisfied.

- algorithms: backtracking search + inference
  (e.g., forward checking, arc consistency, path consistency)
- variable and value orders important
- more efficient: exploit structure of constraint graph
  (connected components; trees)

More Advanced Topics

more advanced topics (not considered in this course):

- backjumping: backtracking over several layers
- no-good learning: infer additional constraints based on information collected during backtracking
- local search methods in the space of total, but not necessarily consistent assignments
- tractable constraint classes: identification of constraint types that allow for polynomial algorithms
- solutions of different quality: constraint optimization problems (COP)

$\rightsquigarrow$ more than enough content for a one-semester course