Foundations of Artificial Intelligence

D6. Constraint Satisfaction Problems: Constraint Graphs

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
- D6–D7. Problem Structure
  - D6. Constraint Graphs
  - D7. Decomposition Methods

D6.1 Constraint Graphs

Motivation

- To solve a constraint network consisting of n variables and k values, k^n assignments must be considered.
- Inference can alleviate this combinatorial explosion, but will not always avoid it.
- Many practically relevant constraint networks are efficiently solvable if their structure is taken into account.

Constraint Graphs

Definition (constraint graph)

Let $\mathcal{C} = \langle V, \mathrm{dom}, (R_{uv}) \rangle$ be a constraint network.

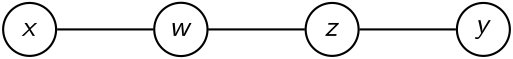
The constraint graph of $\mathcal{C}$ is the graph whose vertices are V and which contains an edge $\{u, v\}$ iff R_{uv} is a nontrivial constraint.

Constraint Graphs: Running Example

Nontrivial Constraints of Running Example

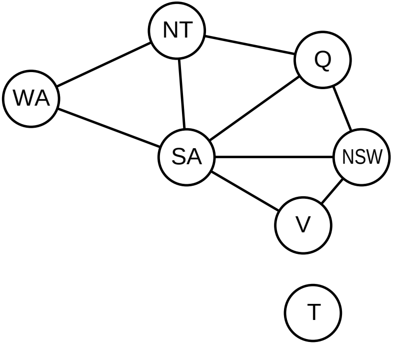
$R_{wx} = \{\langle 2,1\rangle, \langle 4,2\rangle\}$
$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

Resulting Constraint Graph:



Constraint Graphs: Better Example

Coloring of the Australian states (plus Northern Territory)



D6.2 Unconnected Graphs

Unconnected Constraint Graphs

Proposition (unconnected constraint graphs)

If the constraint graph of $\mathcal{C}$ has multiple connected components, the subproblems induced by each component can be solved separately.

The union of the solutions of these subproblems is a solution for $\mathcal{C}$.

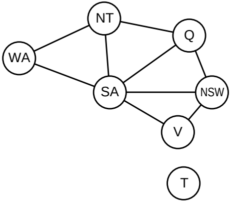
Proof.

A total assignment consisting of combined subsolutions satisfies all constraints that occur within the subproblems.

All constraints between two subproblems are trivial (follows from the definitions of constraint graphs and connected components). □

Unconnected Constraint Graphs: Example

example: Tasmania can be colored independently from the rest of Australia.



further example:
network with $k = 2$, $n = 30$ that decomposes into three components of equal size

savings?
only $3 \cdot 2^{10} = 3072$ assignments instead of $2^{30} = 1073741824$

D6.3 Trees

Trees as Constraint Graphs

Proposition (trees as constraint graphs)

Let $\mathcal{C}$ be a constraint network with n variables and maximal domain size k whose constraint graph is a tree or forest (i.e., does not contain cycles).

Then we can solve $\mathcal{C}$ or prove that no solution exists in time $O(nk^2)$.

example: $k = 5, n = 10$
$\rightsquigarrow k^n = 9765625,\ nk^2 = 250$

Trees as Constraint Graphs: Algorithm

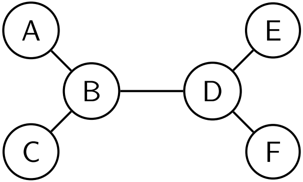
algorithm for trees:

- Build a directed tree for the constraint graph.
  Select an arbitrary variable as the root.
- Order variables $v_1, \dots, v_n$ such that parents are ordered before their children.
- For $i \in \langle n, n-1, \dots, 2 \rangle$: call revise($v_{\text{parent}(i)}, v_i$)
  $\rightsquigarrow$ each variable is arc consistent with respect to its children
- If a domain becomes empty, the problem is unsolvable.
- Otherwise: solve with BacktrackingWithInference, variable order $v_1, \dots, v_n$ and forward checking.
  $\rightsquigarrow$ solution is found without backtracking steps

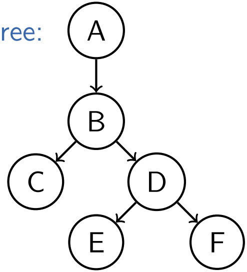
proof: $\rightsquigarrow$ exercises

Trees as Constraint Graphs: Example

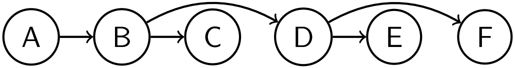
1. constraint graph:



2. directed tree:



3. order:



4. revise steps:

- revise(D, F)
- revise(D, E)
- revise(B, D)
- revise(B, C)
- revise(A, B)

5. finding a solution:

backtracking with forward checking and order $A \prec B \prec C \prec D \prec E \prec F$

D6.4 Summary

Summary

- Constraint networks with simple structure are easy to solve.
- Constraint graphs formalize this structure:
  - several connected components:
    solve separately for each component
  - tree: algorithm linear in number of variables