Foundations of Artificial Intelligence

D5. Constraint Satisfaction Problems: Path Consistency

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
  - D3. Backtracking
  - D4. Arc Consistency
  - D5. Path Consistency
- D6–D7. Problem Structure

Beyond Arc Consistency

Beyond Arc Consistency: Path Consistency

idea of arc consistency:

- For every assignment to a variable u there must be a suitable assignment to every other variable v.
- If not: remove values of u for which no suitable "partner" assignment to v exists.

$\leadsto$ tighter unary constraint on u

This idea can be extended to three variables (path consistency):

- For every joint assignment to variables u, v there must be a suitable assignment to every third variable w.
- If not: remove pairs of values of u and v for which no suitable "partner" assignment to w exists.

$\leadsto$ tighter binary constraint on u and v

German: Pfadkonsistenz

Beyond Arc Consistency: i-Consistency

general concept of i-consistency for $i \geq 2$:

- For every joint assignment to variables $v_1, \dots, v_{i-1}$ there must be a suitable assignment to every i-th variable v_i.
- If not: remove value tuples of $v_1, \dots, v_{i-1}$ for which no suitable "partner" assignment for v_i exists.

$\rightsquigarrow$ tighter $(i-1)$-ary constraint on $v_1, \dots, v_{i-1}$

- 2-consistency = arc consistency
- 3-consistency = path consistency (*)

We do not consider general i-consistency further
as larger values than $i = 3$ are rarely used
and we restrict ourselves to binary constraints in this course.

(*) usual definitions of 3-consistency vs. path consistency differ
when ternary constraints are allowed

Path Consistency

Path Consistency: Definition

Definition (path consistent)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.

1. Two different variables $u, v \in V$ are path consistent with respect to a third variable $w \in V$ if for all values $d_u \in \text{dom}(u), d_v \in \text{dom}(v)$ with $\langle d_u, d_v \rangle \in R_{uv}$ there is a value $d_w \in \text{dom}(w)$ with $\langle d_u, d_w \rangle \in R_{uw}$ and $\langle d_v, d_w \rangle \in R_{vw}$.
2. The constraint network $\mathcal{C}$ is path consistent if for all triples of different variables u, v, w, the variables u and v are path consistent with respect to w.

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

Are w and y path consistent with respect to z?

Path Consistency on Running Example

Running Example

$$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$$
$$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$$
$$\begin{aligned} R_{wy} = \{&\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \\ &\langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \\ &\langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \\ &\langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\} \end{aligned}$$

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Path Consistency on Running Example

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Are w and y path consistent with respect to z? No!

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$
$R_{wy} = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$

Are w and y path consistent with respect to z?

Path Consistency on Running Example

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$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
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Are w and y path consistent with respect to z? Yes!

Path Consistency on Running Example

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Are w and y path consistent with respect to z? Yes!

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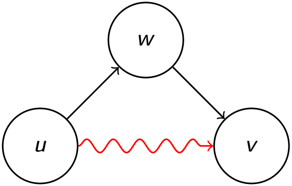
Are w and y path consistent with respect to z? Yes!

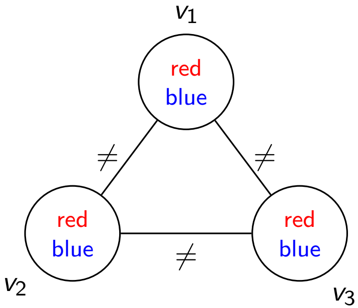
Path Consistency: Remarks

remarks:

- Even if the constraint R_{uv} is trivial, path consistency can infer nontrivial constraints between u and v.
- name "path consistency":
  path $u \rightarrow w \rightarrow v$ leads to new information on $u \rightarrow v$



Path Consistency: Example



arc consistent, but not path consistent

Processing Variable Triples: revise-3

analogous to revise for arc consistency:

function revise-3($\mathcal{C}, u, v, w$):

$\langle V, \mathrm{dom}, (R_{uv}) \rangle := \mathcal{C}$

for each $\langle d_u, d_v \rangle \in R_{uv}$:

  if there is no $d_w \in \mathrm{dom}(w)$ with

   $\langle d_u, d_w \rangle \in R_{uw}$ **and** $\langle d_v, d_w \rangle \in R_{vw}$:

    remove $\langle d_u, d_v \rangle$ from R_{uv}

input: constraint network $\mathcal{C}$ and three variables u, v, w of $\mathcal{C}$

effect: u, v path consistent with respect to w.
All violating pairs are removed from R_{uv}.

time complexity: $O(k^3)$ where k is maximal domain size

Enforcing Path Consistency: PC-2

analogous to AC-3 for arc consistency:

function PC-2($\mathcal{C}$):

$\langle V, \mathrm{dom}, (R_{uv}) \rangle := \mathcal{C}$
queue $:= \emptyset$
for each set of two variables $\{u, v\}$:
    for each $w \in V \setminus \{u, v\}$:
        insert $\langle u, v, w \rangle$ into *queue*
while *queue* $\neq \emptyset$:
    remove any element $\langle u, v, w \rangle$ from *queue*
    revise-3($\mathcal{C}, u, v, w$)
    if R_{uv} changed in the call to revise-3:
        for each $w' \in V \setminus \{u, v\}$:
            insert $\langle w', u, v \rangle$ into *queue*
            insert $\langle w', v, u \rangle$ into *queue*

PC-2: Discussion

The comments for AC-3 hold analogously.

- PC-2 enforces path consistency
- proof idea: invariant of the **while** loop:
  if $\langle u, v, w \rangle \notin \mathit{queue}$, then u, v path consistent with respect to w
- time complexity $O(n^3 k^5)$ for n variables and maximal domain size k (Why?)

Summary

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- generalization of
  arc consistency (considers pairs of variables)
  to path consistency (considers triples of variables)
  and i-consistency (considers i-tuples of variables)
- arc consistency tightens unary constraints
- path consistency tightens binary constraints
- i-consistency tightens $(i-1)$-ary constraints
- higher levels of consistency more powerful
  but more expensive than arc consistency