Foundations of Artificial Intelligence

D5. Constraint Satisfaction Problems: Path Consistency

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
  - D3. Backtracking
  - D4. Arc Consistency
  - D5. Path Consistency
- D6–D7. Problem Structure

D5.1 Beyond Arc Consistency

Beyond Arc Consistency: Path Consistency

idea of arc consistency:

- For every assignment to a variable u there must be a suitable assignment to every other variable v.
- If not: remove values of u for which no suitable "partner" assignment to v exists.

$\rightsquigarrow$ tighter unary constraint on u

This idea can be extended to three variables (path consistency):

- For every joint assignment to variables u, v there must be a suitable assignment to every third variable w.
- If not: remove pairs of values of u and v for which no suitable "partner" assignment to w exists.

$\rightsquigarrow$ tighter binary constraint on u and v

German: Pfadkonsistenz

Beyond Arc Consistency: i-Consistency

general concept of i-consistency for $i \geq 2$:

- For every joint assignment to variables $v_1, \dots, v_{i-1}$ there must be a suitable assignment to every i-th variable v_i.
- If not: remove value tuples of $v_1, \dots, v_{i-1}$ for which no suitable "partner" assignment for v_i exists.

$\rightsquigarrow$ tighter $(i-1)$-ary constraint on $v_1, \dots, v_{i-1}$

- 2-consistency = arc consistency
- 3-consistency = path consistency (*)

We do not consider general i-consistency further as larger values than $i = 3$ are rarely used and we restrict ourselves to binary constraints in this course.

(*) usual definitions of 3-consistency vs. path consistency differ when ternary constraints are allowed

D5.2 Path Consistency

Path Consistency: Definition

Definition (path consistent)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.

1. Two different variables $u, v \in V$ are path consistent with respect to a third variable $w \in V$ if for all values $d_u \in \text{dom}(u), d_v \in \text{dom}(v)$ with $\langle d_u, d_v \rangle \in R_{uv}$ there is a value $d_w \in \text{dom}(w)$ with $\langle d_u, d_w \rangle \in R_{uw}$ and $\langle d_v, d_w \rangle \in R_{vw}$.
2. The constraint network $\mathcal{C}$ is path consistent if for all triples of different variables u, v, w, the variables u and v are path consistent with respect to w.

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$
$R_{wy} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$

Are w and y path consistent with respect to z? No!

Path Consistency on Running Example

Running Example

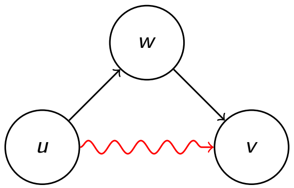
$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$
$R_{wy} = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$

Are w and y path consistent with respect to z? Yes!

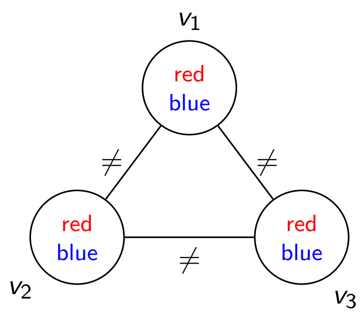
Path Consistency: Remarks

remarks:

- Even if the constraint R_{uv} is trivial, path consistency can infer nontrivial constraints between u and v.
- name "path consistency": path $u \to w \to v$ leads to new information on $u \to v$



Path Consistency: Example



arc consistent, but not path consistent

Processing Variable Triples: revise-3

analogous to revise for arc consistency:

function revise-3($\mathcal{C}, u, v, w$):
$\langle V, \text{dom}, (R_{uv})\rangle := \mathcal{C}$
for each $\langle d_u, d_v\rangle \in R_{uv}$:
  if there is no $d_w \in \text{dom}(w)$ with
   $\langle d_u, d_w\rangle \in R_{uw}$ **and** $\langle d_v, d_w\rangle \in R_{vw}$:
    remove $\langle d_u, d_v\rangle$ from R_{uv}

input: constraint network $\mathcal{C}$ and three variables u, v, w of $\mathcal{C}$

effect: u, v path consistent with respect to w.
All violating pairs are removed from R_{uv}.

time complexity: $O(k^3)$ where k is maximal domain size

Enforcing Path Consistency: PC-2

analogous to AC-3 for arc consistency:

function PC-2($\mathcal{C}$):
$\langle V, \text{dom}, (R_{uv})\rangle := \mathcal{C}$
queue $:= \emptyset$
for each set of two variables $\{u, v\}$:
  for each $w \in V \setminus \{u, v\}$:
    insert $\langle u, v, w\rangle$ into *queue*
while *queue* $\neq \emptyset$:
  remove any element $\langle u, v, w\rangle$ from *queue*
  revise-3($\mathcal{C}, u, v, w$)
  if R_{uv} changed in the call to revise-3:
    for each $w' \in V \setminus \{u, v\}$:
      insert $\langle w', u, v\rangle$ into *queue*
      insert $\langle w', v, u\rangle$ into *queue*

PC-2: Discussion

The comments for AC-3 hold analogously.

- PC-2 enforces path consistency
- proof idea: invariant of the **while** loop:
  if $\langle u, v, w\rangle \notin$ *queue*, then u, v path consistent with respect to w
- time complexity $O(n^3k^5)$ for n variables and maximal domain size k (Why?)

D5.3 Summary

Summary

- generalization of
  arc consistency (considers pairs of variables)
  to path consistency (considers triples of variables)
  and i-consistency (considers i-tuples of variables)
- arc consistency tightens unary constraints
- path consistency tightens binary constraints
- i-consistency tightens $(i-1)$-ary constraints
- higher levels of consistency more powerful
  but more expensive than arc consistency