Foundations of Artificial Intelligence

D4. Constraint Satisfaction Problems: Arc Consistency

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
  - D3. Backtracking
  - D4. Arc Consistency
  - D5. Path Consistency
- D6–D7. Problem Structure

Inference

Inference

Inference

Derive additional constraints (here: unary or binary)
that are implied by the given constraints,
i.e., that are satisfied in all solutions.

Inference: Example

Running Example

binary constraints:

- $R_{wx} = \{\langle 2, 1\rangle, \langle 4, 2\rangle\}$
- $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
- $R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

domains:

- $\text{dom}(w) = \{1, 2, 3, 4\}$
- $\text{dom}(x) = \{1, 2, 3\}$
- $\text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(z) = \{1, 2, 3\}$

Can we use the constraint R_{wz} $(w < z)$ to come up with a unary constraint R_w?

Inference: Example

Running Example

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domains (unary constraints):

- $\text{dom}(w) = \{1, 2\}$
- $\text{dom}(x) = \{1, 2, 3\}$
- $\text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(z) = \{1, 2, 3\}$

Can we use the constraint R_{wz} $(w < z)$ to come up with a unary constraint R_w?

$\rightsquigarrow$ tighten domain with unary constraint
(sometimes called node consistency)

Inference: Example

Running Example

binary constraints:

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- $R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
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domains (unary constraints):

- $\mathrm{dom}(w) = \{1,2\}$
- $\mathrm{dom}(x) = \{1,2,3\}$
- $\mathrm{dom}(y) = \{1,2,3,4\}$
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How does this affect the binary constraint R_{wx}?

Inference: Example

Running Example

binary constraints:

- $R_{wx} = \{\langle 2,1\rangle\}$
- $R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
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How does this affect the binary constraint R_{wx}?

Inference: Example

Running Example

binary constraints:

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domains (unary constraints):

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- $\mathrm{dom}(x) = \{1, 2, 3\}$
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- $\mathrm{dom}(z) = \{1, 2, 3\}$

Can we generate a "new" binary constraint from $w < z$ and $z < y$? (i.e., tighten a trivial constraint)

Inference: Example

Running Example

binary constraints:

- $R_{wx} = \{\langle 2,1 \rangle\}$
- $R_{wz} = \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,3 \rangle\}$
- $R_{yz} = \{\langle 2,1 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle, \langle 4,1 \rangle, \langle 4,2 \rangle, \langle 4,3 \rangle\}$
- $R_{wy} = \{\langle 1,3 \rangle, \langle 1,4 \rangle, \langle 2,4 \rangle\}$

domains (unary constraints):

- $\text{dom}(w) = \{1,2\}$
- $\text{dom}(x) = \{1,2,3\}$
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Can we generate a "new" binary constraint from $w < z$ and $z < y$?
(i.e., tighten a trivial constraint)

Trade-Off Search vs. Inference

Inference formally

For a given constraint network $\mathcal{C}$, replace $\mathcal{C}$ with an equivalent, but tighter constraint network.

Trade-off:

- the more complex the inference, and
- the more often inference is applied,
- the smaller the resulting state space, but
- the higher the complexity per search node.

When to Apply Inference?

different possibilities to apply inference:

- once as preprocessing before search
- combined with search: before recursive calls during backtracking procedure
  - already assigned variable $v \mapsto d$ corresponds to $\text{dom}(v) = \{d\}$ $\rightsquigarrow$ more inferences possible
  - during backtracking, derived constraints have to be retracted because they were based on the given assignment

  $\rightsquigarrow$ powerful, but possibly expensive

Backtracking with Inference

function BacktrackingWithInference($\mathcal{C}, \alpha$):

if α is inconsistent with $\mathcal{C}$:
    return inconsistent

if α is a total assignment:
    return α

$\mathcal{C}' := \langle V, \mathrm{dom}', (R'_{uv}) \rangle :=$ copy of $\mathcal{C}$
apply inference to $\mathcal{C}'$

if $\mathrm{dom}'(v) \neq \emptyset$ for all variables v:

    select some variable v for which α is not defined

    for each $d \in$ copy of $\mathrm{dom}'(v)$ in some order:
        $\alpha' := \alpha \cup \{v \mapsto d\}$
        $\mathrm{dom}'(v) := \{d\}$
        $\alpha'' :=$ BacktrackingWithInference($\mathcal{C}', \alpha'$)
        if $\alpha'' \neq$ **inconsistent**:
            return α''

return inconsistent

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        $\alpha' := \alpha \cup \{v \mapsto d\}$
        $\mathrm{dom}'(v) := \{d\}$
        $\alpha'' :=$ BacktrackingWithInference($\mathcal{C}', \alpha'$)
        if $\alpha'' \neq$ **inconsistent**:
            return α''

return inconsistent

Backtracking with Inference: Discussion

- Inference is a placeholder:
  different inference methods can be applied.
- Inference methods can recognize unsolvability (given α)
  and indicate this by clearing the domain of a variable.
- Efficient implementations of inference are often incremental:
  the last assigned variable/value pair $v \mapsto d$ is taken
  into account to speed up the inference computation.

Forward Checking

Forward Checking

We start with a simple inference method:

Forward Checking

Let α be a partial assignment.
Inference: For all unassigned variables v in α,
remove all values from the domain of v that are in conflict
with already assigned variable/value pairs in α.

$\rightsquigarrow$ definition of conflict as in the previous chapter

Incremental computation:

- When adding $v \mapsto d$ to the assignment,
  delete all pairs that conflict with $v \mapsto d$.

Forward Checking: Example

Running Example

Removing values in conflict with $\alpha = \{w \mapsto 2\}$:

binary constraints:

- $R_{wx} = \{\langle 2, 1\rangle, \langle 4, 2\rangle\}$
- $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
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domains:

- w is already assigned
- $\mathrm{dom}(x) = \{1, 2, 3\}$
- $\mathrm{dom}(y) = \{1, 2, 3, 4\}$
- $\mathrm{dom}(z) = \{1, 2, 3\}$

Forward Checking: Example

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domains:

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Forward Checking: Example

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domains:

- w is already assigned
- $\text{dom}(x) = \{1\}$
- $\text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(z) = \{1, 2, 3\}$

Forward Checking: Example

Running Example

Removing values in conflict with $\alpha = \{w \mapsto 2\}$:

binary constraints:

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- $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
- $R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

domains:

- w is already assigned
- $\text{dom}(x) = \{1\}$
- $\text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(z) = \{3\}$

Forward Checking: Discussion

properties of forward checking:

- correct inference method (retains equivalence)
- affects domains (= unary constraints), but not binary constraints
- consistency check at the beginning of the backtracking procedure no longer needed (Why?)
- cheap, but often still useful inference method

⇝ apply at least forward checking in the backtracking procedure

In the following, we will consider more powerful inference methods.

Arc Consistency

Arc Consistency: Definition

Definition (Arc Consistent)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.

1. The variable $v \in V$ is arc consistent with respect to another variable $v' \in V$, if for every value $d \in \text{dom}(v)$ there exists a value $d' \in \text{dom}(v')$ with $\langle d, d' \rangle \in R_{vv'}$.
2. The constraint network $\mathcal{C}$ is arc consistent, if every variable $v \in V$ is arc consistent with respect to every other variable $v' \in V$.

German: kantenkonsistent

remarks:

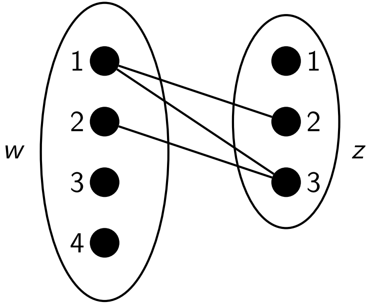
- definition for variable pair is not symmetrical
- v always arc consistent with respect to v' if the constraint between v and v' is trivial

Arc Consistency: Example

Running Example

Consider variables w and z from our running example:

- $\text{dom}(w) = \{1, 2, 3, 4\}$
- $\text{dom}(z) = \{1, 2, 3\}$
- $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$



Arc consistency
of w with respect to z and
of z with respect to w
is violated.

Enforcing Arc Consistency

- Enforcing arc consistency, i.e., removing values from dom(v) that violate the arc consistency of v with respect to v', is a correct inference method. (Why?)
- more powerful than forward checking (Why?)

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- Enforcing arc consistency, i.e., removing values from dom(v) that violate the arc consistency of v with respect to v', is a correct inference method. (Why?)
- more powerful than forward checking (Why?)
  - ⇝ Forward checking is a special case: enforcing arc consistency of all variables with respect to the just assigned variable corresponds to forward checking.

We will next consider algorithms that enforce arc consistency.

Processing Variable Pairs: revise

function revise($\mathcal{C}, v, v'$):

$\langle V, \text{dom}, (R_{uv})\rangle := \mathcal{C}$

for each $d \in \text{dom}(v)$:

  if there is no $d' \in \text{dom}(v')$ with $\langle d, d'\rangle \in R_{vv'}$:

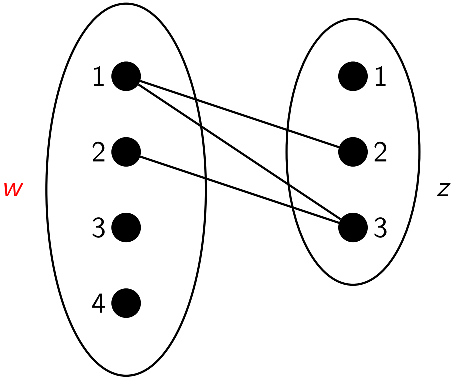
    remove d from $\text{dom}(v)$

input: constraint network $\mathcal{C}$ and two variables v, v' of $\mathcal{C}$

effect: v arc consistent with respect to v'.
All violating values in $\text{dom}(v)$ are removed.

time complexity: $O(k^2)$, where k is maximal domain size

revise($\mathcal{C}$, w, z) in Running Example



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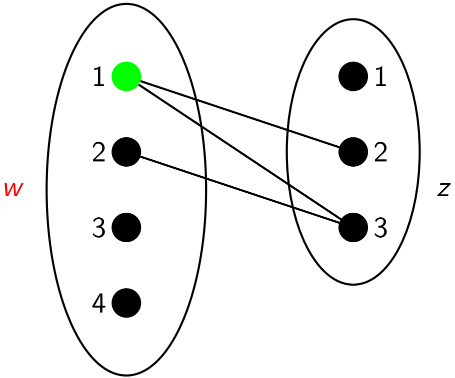

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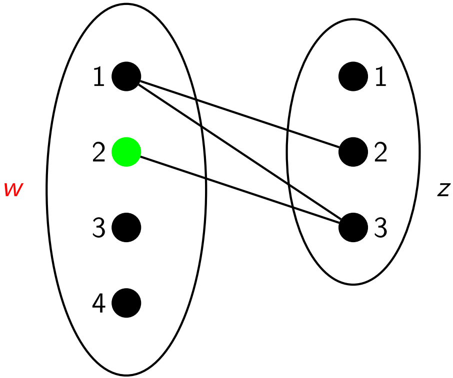

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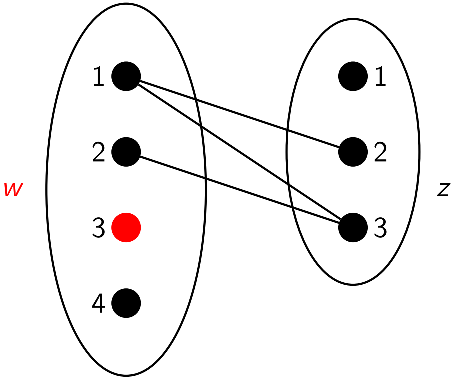

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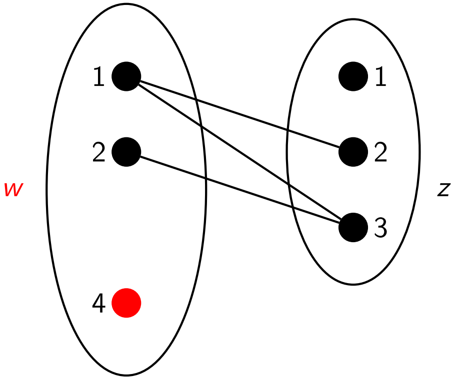

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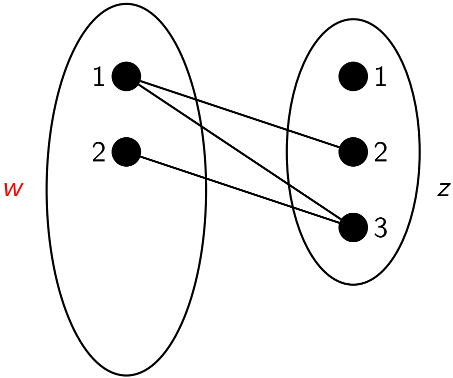

Enforcing Arc Consistency: AC-1

function AC-1($\mathcal{C}$):

$\langle V, \mathrm{dom}, (R_{uv}) \rangle := \mathcal{C}$

repeat

  for each nontrivial constraint R_{uv}:

    revise($\mathcal{C}$, u, v)

    revise($\mathcal{C}$, v, u)

until no domain has changed in this iteration

input: constraint network $\mathcal{C}$

effect: transforms $\mathcal{C}$ into equivalent arc consistent network

time complexity: ?

Enforcing Arc Consistency: AC-1

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effect: transforms $\mathcal{C}$ into equivalent arc consistent network

time complexity: $O(n \cdot e \cdot k^3)$, with n variables, e nontrivial constraints and maximal domain size k

AC-1: Discussion

- AC-1 does the job, but is rather inefficient.
- Drawback: Variable pairs are often checked again and again although their domains have remained unchanged.
- These (redundant) checks can be saved.

$\rightsquigarrow$ more efficient algorithm: AC-3

Enforcing Arc Consistency: AC-3

idea: store potentially inconsistent variable pairs in a queue

function AC-3($\mathcal{C}$):

$\langle V, \mathrm{dom}, (R_{uv}) \rangle := \mathcal{C}$

$queue := \emptyset$

for each nontrivial constraint R_{uv}:

  insert $\langle u, v \rangle$ into *queue*

  insert $\langle v, u \rangle$ into *queue*

while $queue \neq \emptyset$:

  remove an arbitrary element $\langle u, v \rangle$ from *queue*

  revise($\mathcal{C}, u, v$)

  if dom(u) changed in the call to revise:

    for each $w \in V \setminus \{u, v\}$ where R_{wu} is nontrivial:

      insert $\langle w, u \rangle$ into *queue*

AC-3: Discussion

- *queue* can be an arbitrary data structure that supports insert and remove operations (the order of removal does not affect the result)

$\rightsquigarrow$ use data structure with fast insertion and removal, e.g., stack

- AC-3 has the same effect as AC-1: it enforces arc consistency
- proof idea: invariant of the **while** loop: If $\langle u, v \rangle \notin$ *queue*, then u is arc consistent with respect to v

AC-3: Time Complexity

Proposition (time complexity of AC-3)

Let $\mathcal{C}$ be a constraint network with e nontrivial constraints and maximal domain size k.

The time complexity of AC-3 *is $O(e \cdot k^3)$.*

AC-3: Time Complexity (Proof)

Proof.

Consider a pair $\langle u, v\rangle$ such that there exists a nontrivial constraint R_{uv} or R_{vu}. (There are at most $2e$ of such pairs.)

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Every time this pair is inserted to the queue (except for the first time) the domain of the second variable has just been reduced.

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Hence every pair $\langle u, v\rangle$ is inserted into the queue at most $k + 1$ times $\rightsquigarrow$ at most $O(ek)$ insert operations in total.

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This can happen at most k times.

Hence every pair $\langle u, v \rangle$ is inserted into the queue at most $k + 1$ times $\rightsquigarrow$ at most $O(ek)$ insert operations in total.

This bounds the number of **while** iterations by $O(ek)$, giving an overall time complexity of $O(ek) \cdot O(k^2) = O(ek^3)$. □

Summary

Summary: Inference

- inference: derivation of additional constraints that are implied by the known constraints

$\rightsquigarrow$ tighter equivalent constraint network

- trade-off search vs. inference
- inference as preprocessing or integrated into backtracking

Summary: Forward Checking, Arc Consistency

- cheap and easy inference: forward checking
  - remove values that conflict with already assigned values
- more expensive and more powerful: arc consistency
  - iteratively remove values without a suitable "partner value" for another variable until fixed-point reached
  - efficient implementation of AC-3: $O(ek^3)$ with e: #nontrivial constraints, k: size of domain