

# Foundations of Artificial Intelligence

## D2. Constraint Satisfaction Problems: Constraint Networks

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# Constraint Satisfaction Problems: Overview

## Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
  - D1. Introduction and Examples
  - D2. Constraint Networks
- D3–D5. Basic Algorithms
- D6–D7. Problem Structure

# Constraint Networks

# Constraint Networks: Informally

## Constraint Networks: Informal Definition

A **constraint network** is defined by

- a finite set of **variables**
- a finite **domain** for each variable
- a set of **constraints** (here: **binary relations**)

The objective is to find a **solution** for the constraint network, i.e., an assignment of the variables that complies with all constraints.

Informally, people often just speak of **constraint satisfaction problems (CSP)** instead of constraint networks.

More formally, a “CSP” is the algorithmic problem of finding a solution for a constraint network.

# Constraint Networks: Formally

## Definition (binary constraint network)

A **(binary) constraint network**

is a 3-tuple  $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$  such that:

- $V$  is a non-empty and finite set of **variables**,
- $\text{dom}$  is a function that assigns a non-empty and finite **domain** to each variable  $v \in V$ , and
- $(R_{uv})_{u,v \in V, u \neq v}$  is a family of binary relations (**constraints**) over  $V$  where for all  $u \neq v$ :  $R_{uv} \subseteq \text{dom}(u) \times \text{dom}(v)$

**German:** (binäres) Constraint-Netz, Variablen, Wertebereich, Constraints

**possible generalizations:**

- infinite domains (e.g.,  $\text{dom}(v) = \mathbb{Z}$ )
- constraints of higher arity  
(e.g., satisfiability in propositional logic)

# Variables and Domains

## Running Example (informally)

- assign a value from  $\{1, 2, 3, 4\}$  to the variables  $w$  and  $y$
- and from  $\{1, 2, 3\}$  to  $x$  and  $z$
- such that ...

## Running Example (formally)

$\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$  with

- $V = \{w, x, y, z\}$
- $\text{dom}(w) = \text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(x) = \text{dom}(z) = \{1, 2, 3\}$
- ...

# Binary Constraints (1)

binary constraints:

- For variables  $u, v$ , the constraint  $R_{uv}$  expresses which **joint assignments** to  $u$  and  $v$  are allowed in a solution.

Running Example (informally)

- ... such that
  - ...,  $w < z$ , ...

Running Example (formally)

...,  $R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \dots$

## Binary Constraints (2)

### binary constraints:

- If  $R_{uv} = \text{dom}(u) \times \text{dom}(v)$ , the constraint is **trivial**: there is no restriction, and the constraint is typically not given explicitly in the constraint network description (although it formally always exists!).

### Running Example

$$\begin{aligned} \dots, R_{xz} = \{ & \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \\ & \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \\ & \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \}, \dots \end{aligned}$$



## Binary Constraints (3)

binary constraints:

- Constraints  $R_{uv}$  and  $R_{vu}$  refer to the same variables.  
Hence, usually only one of them is given in the description.

### Running Example (informally)

- ... such that
  - ...,  $w < z$ , ...

### Running Example (formally)

$$\dots, R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \dots$$

$$\dots, R_{zw} = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}, \dots$$

# Unary Constraints

## unary constraints:

- It is often useful to have additional restrictions on **single** variables as constraints.
- Such constraints are called **unary** constraints.
- A unary constraint  $R_v$  for  $v \in V$  corresponds to a restriction of  $\text{dom}(v)$  to the values allowed by  $R_v$ .
- Formally, unary constraints are not necessary, but they often allow us to describe constraint networks more clearly.

**German:** unäre Constraints

### Running Example

$\text{dom}(z) = \{1, 2, 3\}$  could be described as  
 $\text{dom}(z) = \{1, 2, 3, 4\}, R_z = \{1, 2, 3\}$

# Example

## Full Formal Model of Running Example

$\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$  with

- variables:

$$V = \{w, x, y, z\}$$

- domains:

$$\text{dom}(w) = \text{dom}(y) = \{1, 2, 3, 4\}$$

$$\text{dom}(x) = \text{dom}(z) = \{1, 2, 3\}$$

- constraints:

$$R_{wx} = \{\langle 2, 1 \rangle, \langle 4, 2 \rangle\}$$

$$R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$$

$$R_{yz} = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \\ \langle 4, 1 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle\}$$

# Compact Encodings and General Constraint Solvers

Constraint networks allow for **compact encodings** of large sets of assignments:

- Consider a network with  $n$  variables with domains of size  $k$ .

↪  $k^n$  assignments

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~>  $k^n$  assignments

- For the **description** as constraint network, at most  $\binom{n}{2}$ , i.e.,  $O(n^2)$  constraints have to be provided.

Every constraint in turn consists of at most  $O(k^2)$  pairs.

~> encoding size  $O(n^2 k^2)$

- We observe: The number of assignments is **exponentially larger** than the description of the constraint network.

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- We observe: The number of assignments is **exponentially larger** than the description of the constraint network.
- As a consequence, such descriptions can be used as inputs of **general** constraint solvers.

# Examples

# Example: 4 Queens Problem

## 4 Queens Problem as Constraint Network

- **variables:**  $V = \{v_1, v_2, v_3, v_4\}$   
 $v_i$  encodes the rank of the queen in the  $i$ -th file
- **domains:**  
 $\text{dom}(v_1) = \text{dom}(v_2) = \text{dom}(v_3) = \text{dom}(v_4) = \{1, 2, 3, 4\}$
- **constraints:** for all  $1 \leq i < j \leq 4$ , we set:  $R_{v_i, v_j} = \{\langle k, l \rangle \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \mid k \neq l \wedge |k - l| \neq |i - j|\}$   
e.g.  $R_{v_1, v_3} = \{\langle 1, 2 \rangle, \langle 1, 4 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 4 \rangle, \langle 4, 1 \rangle, \langle 4, 3 \rangle\}$

	$v_1$	$v_2$	$v_3$	$v_4$
1				
2				
3				
4				



# Example: Sudoku

## Sudoku as Constraint Network

- **variables:**  $V = \{v_{ij} \mid 1 \leq i, j \leq 9\}$ ;  $v_{ij}$ : Value row  $i$ , column  $j$
- **domains:**  $\text{dom}(v) = \{1, \dots, 9\}$  for all  $v \in V$
- **unary constraints:**  $R_{v_{ij}} = \{k\}$ ,  
if  $\langle i, j \rangle$  is a cell with predefined value  $k$
- **binary constraints:** for all  $v_{ij}, v_{i'j'} \in V$ , we set  
 $R_{v_{ij}v_{i'j'}} = \{\langle a, b \rangle \in \{1, \dots, 9\} \times \{1, \dots, 9\} \mid a \neq b\}$ ,  
if  $i = i'$  (same row), or  $j = j'$  (same column),  
or  $\langle \lceil \frac{i}{3} \rceil, \lceil \frac{j}{3} \rceil \rangle = \langle \lceil \frac{i'}{3} \rceil, \lceil \frac{j'}{3} \rceil \rangle$  (same block)

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

# Assignments and Consistency

# Assignments

## Definition (assignment, partial assignment)

Let  $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$  be a constraint network.

A **partial assignment** of  $\mathcal{C}$  (or of  $V$ ) is a function

$$\alpha : V' \rightarrow \bigcup_{v \in V} \text{dom}(v)$$

with  $V' \subseteq V$  and  $\alpha(v) \in \text{dom}(v)$  for all  $v \in V'$ .

If  $V' = V$ , then  $\alpha$  is also called **total assignment** (or **assignment**).

**German:** partielle Belegung, (totale) Belegung

~> **partial assignments** assign values to some or to all variables

~> (total) **assignments** are defined on all variables

# Example

## Partial Assignments of Running Example

$$\alpha_1 = \{w \mapsto 1, z \mapsto 2\}$$

$$\alpha_2 = \{w \mapsto 3, x \mapsto 1\}$$

## Total Assignments of Running Example

$$\alpha_3 = \{w \mapsto 1, x \mapsto 1, y \mapsto 2, z \mapsto 2\}$$

$$\alpha_4 = \{w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3\}$$

# Consistency

## Definition (inconsistent, consistent, violated)

A partial assignment  $\alpha$  of a constraint network  $\mathcal{C}$  is called **inconsistent** if there are variables  $u, v$  such that  $\alpha$  is defined for both  $u$  and  $v$ , and  $\langle \alpha(u), \alpha(v) \rangle \notin R_{uv}$ .

In this case, we say  $\alpha$  **violates** the constraint  $R_{uv}$ .

A partial assignment is called **consistent** if it is not inconsistent.

**German:** inkonsistent, verletzt, konsistent

**trivial example:** The empty assignment is always consistent.

# Example

## Consistent Partial Assignment

$$\alpha_1 = \{w \mapsto 1, z \mapsto 2\}$$

## Inconsistent Partial Assignment

$$\alpha_2 = \{w \mapsto 2, x \mapsto 2\}$$

violates  $R_{wx} = \{\langle 2, 1 \rangle, \langle 4, 2 \rangle\}$

## Inconsistent Assignment

$$\alpha_3 = \{w \mapsto 2, x \mapsto 1, y \mapsto 2, z \mapsto 2\}$$

violates  $R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$  and

$$R_{yz} = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle\}$$

# Solution

## Definition (solution, solvable)

Let  $\mathcal{C}$  be a constraint network.

A consistent and total assignment of  $\mathcal{C}$  is called a **solution** of  $\mathcal{C}$ .

If a solution of  $\mathcal{C}$  exists,  $\mathcal{C}$  is called **solvable**.

If no solution exists,  $\mathcal{C}$  is called **inconsistent**.

**German:** Lösung, lösbar, inkonsistent

## Solution of the Running Example

$$\alpha = \{w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3\}$$

# Consistency vs. Solvability

**Note:** Consistent partial assignments  $\alpha$  **cannot necessarily** be extended to a solution.

It only means that **so far** (i.e., on the variables where  $\alpha$  is defined) no constraint is violated.

**Example (4 queens problem):**  $\alpha = \{v_1 \mapsto 1, v_2 \mapsto 4, v_3 \mapsto 2\}$

	$v_1$	$v_2$	$v_3$	$v_4$
1	q			
2			q	
3				
4		q		



# Complexity of Constraint Satisfaction Problems

## Proposition (CSPs are NP-complete)

*It is an NP-complete problem to decide whether a given constraint network is solvable.*

## Proof

### Membership in NP:

Guess and check: guess a solution and check it for validity.  
This can be done in polynomial time in the size of the input.

### NP-hardness:

The graph coloring problem is a special case of CSPs and is already known to be NP-complete.

# Tightness of Constraint Networks

## Definition (tighter, strictly tighter)

Let  $\mathcal{C} = \langle V, \text{dom}, R_{uv} \rangle$  and  $\mathcal{C}' = \langle V, \text{dom}', R'_{uv} \rangle$  be constraint networks with equal variable sets  $V$ .

$\mathcal{C}$  is called **tighter** than  $\mathcal{C}'$ , in symbols  $\mathcal{C} \sqsubseteq \mathcal{C}'$ , if

- $\text{dom}(v) \subseteq \text{dom}'(v)$  for all  $v \in V$ , and
- $R_{uv} \subseteq R'_{uv}$  for all  $u, v \in V$  (including trivial constraints).

If at least one of these subset equations is strict, then  $\mathcal{C}$  is called **strictly tighter** than  $\mathcal{C}'$ , in symbols  $\mathcal{C} \sqsubset \mathcal{C}'$ .

**German:** (echt) schärfer

# Equivalence of Constraint Networks

## Definition (equivalent)

Let  $\mathcal{C}$  and  $\mathcal{C}'$  be constraint networks with equal variable sets.  
 $\mathcal{C}$  and  $\mathcal{C}'$  are called **equivalent**, in symbols  $\mathcal{C} \equiv \mathcal{C}'$ ,  
 if they have the same solutions.

**German:** äquivalent

# Outline and Summary

# CSP Algorithms

In the following chapters, we will consider **solution algorithms** for constraint networks.

basic concepts:

- **search**: check partial assignments systematically
- **backtracking**: discard inconsistent partial assignments
- **inference**: derive equivalent, but tighter constraints to reduce the size of the search space

# Summary

- formal definition of **constraint networks**:  
**variables, domains, constraints**
- **compact encodings** of exponentially many configurations
- **unary** and **binary** constraints
- **assignments**: partial and total
- **consistency** of assignments; **solutions**
- deciding solvability is **NP-complete**
- **tightness** of constraints
- **equivalence** of constraints