Foundations of Artificial Intelligence

D2. Constraint Satisfaction Problems: Constraint Networks

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
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- D3–D5. Basic Algorithms
- D6–D7. Problem Structure

D2.1 Constraint Networks

Constraint Networks: Informally

Constraint Networks: Informal Definition

A constraint network is defined by

- a finite set of variables
- a finite domain for each variable
- a set of constraints (here: binary relations)

The objective is to find a solution for the constraint network, i.e., an assignment of the variables that complies with all constraints.

Informally, people often just speak of constraint satisfaction problems (CSP) instead of constraint networks.

More formally, a "CSP" is the algorithmic problem of finding a solution for a constraint network.

Constraint Networks: Formally

Definition (binary constraint network)

A (binary) constraint network
is a 3-tuple $\mathcal{C} = \langle V, \mathrm{dom}, (R_{uv}) \rangle$ such that:

- V is a non-empty and finite set of variables,
- dom is a function that assigns a non-empty and finite domain to each variable $v \in V$, and
- $(R_{uv})_{u,v \in V, u \neq v}$ is a family of binary relations (constraints) over V where for all $u \neq v$: $R_{uv} \subseteq \mathrm{dom}(u) \times \mathrm{dom}(v)$

German: (binäres) Constraint-Netz, Variablen, Wertebereich, Constraints

possible generalizations:

- infinite domains (e.g., $\mathrm{dom}(v) = \mathbb{Z}$)
- constraints of higher arity
  (e.g., satisfiability in propositional logic)

Variables and Domains

Running Example (informally)

- assign a value from $\{1, 2, 3, 4\}$ to the variables w and y
- and from $\{1, 2, 3\}$ to x and z
- such that ...

Running Example (formally)

$\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ with

- $V = \{w, x, y, z\}$
- $\text{dom}(w) = \text{dom}(y) = \{1, 2, 3, 4\}$
- $\text{dom}(x) = \text{dom}(z) = \{1, 2, 3\}$
- ...

Binary Constraints (1)

binary constraints:

- For variables u, v, the constraint R_{uv} expresses which joint assignments to u and v are allowed in a solution.

Running Example (informally)

- ... such that
  - $\dots, w < z, \dots$

Running Example (formally)

$\dots, R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}, \dots$

Binary Constraints (2)

binary constraints:

- If $R_{uv} = \text{dom}(u) \times \text{dom}(v)$, the constraint is trivial: there is no restriction, and the constraint is typically not given explicitly in the constraint network description (although it formally always exists!).

Running Example

$$\ldots, R_{xz} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}, \ldots$$

Binary Constraints (3)

binary constraints:

- Constraints R_{uv} and R_{vu} refer to the same variables.
  Hence, usually only one of them is given in the description.

Running Example (informally)

- ... such that
  - $\dots, w < z, \dots$

Running Example (formally)

$\dots, R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}, \dots$
$\dots, R_{zw} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}, \dots$

Unary Constraints

unary constraints:

- It is often useful to have additional restrictions on single variables as constraints.
- Such constraints are called unary constraints.
- A unary constraint R_v for $v \in V$ corresponds to a restriction of $\text{dom}(v)$ to the values allowed by R_v.
- Formally, unary constraints are not necessary, but they often allow us to describe constraint networks more clearly.

German: unäre Constraints

Running Example

$\text{dom}(z) = \{1, 2, 3\}$ could be described as
$\text{dom}(z) = \{1, 2, 3, 4\}, R_z = \{1, 2, 3\}$

Example

Full Formal Model of Running Example

$\mathcal{C} = \langle V, \mathrm{dom}, (R_{uv}) \rangle$ with

- variables:
  $V = \{w, x, y, z\}$
- domains:
  $\mathrm{dom}(w) = \mathrm{dom}(y) = \{1, 2, 3, 4\}$
  $\mathrm{dom}(x) = \mathrm{dom}(z) = \{1, 2, 3\}$
- constraints:
  $R_{wx} = \{\langle 2, 1\rangle, \langle 4, 2\rangle\}$
  $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
  $R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

Compact Encodings and General Constraint Solvers

Constraint networks allow for compact encodings of large sets of assignments:

- Consider a network with n variables with domains of size k.

$\rightsquigarrow$ k^n assignments

- For the description as constraint network, at most $\binom{n}{2}$, i.e., $O(n^2)$ constraints have to be provided. Every constraint in turn consists of at most $O(k^2)$ pairs.

$\rightsquigarrow$ encoding size $O(n^2 k^2)$

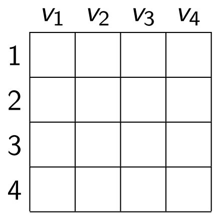
- We observe: The number of assignments is exponentially larger than the description of the constraint network.
- As a consequence, such descriptions can be used as inputs of general constraint solvers.

D2.2 Examples

Example: 4 Queens Problem

4 Queens Problem as Constraint Network

- variables: $V = \{v_1, v_2, v_3, v_4\}$
  v_i encodes the rank of the queen in the i-th file
- domains:
  $\text{dom}(v_1) = \text{dom}(v_2) = \text{dom}(v_3) = \text{dom}(v_4) = \{1, 2, 3, 4\}$
- constraints: for all $1 \leq i < j \leq 4$, we set: $R_{v_i, v_j} = \{\langle k, l \rangle \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \mid k \neq l \wedge |k - l| \neq |i - j|\}$
  e.g. $R_{v_1, v_3} = \{\langle 1, 2\rangle, \langle 1, 4\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle, \langle 3, 4\rangle, \langle 4, 1\rangle, \langle 4, 3\rangle\}$



Example: Sudoku

Sudoku as Constraint Network

- variables: $V = \{v_{ij} \mid 1 \leq i, j \leq 9\}$; v_{ij}: Value row i, column j
- domains: $\mathrm{dom}(v) = \{1, \dots, 9\}$ for all $v \in V$
- unary constraints: $R_{v_{ij}} = \{k\}$,
  if $\langle i, j \rangle$ is a cell with predefined value k
- binary constraints: for all $v_{ij}, v_{i'j'} \in V$, we set
  $R_{v_{ij} v_{i'j'}} = \{\langle a, b \rangle \in \{1, \dots, 9\} \times \{1, \dots, 9\} \mid a \neq b\}$,
  if $i = i'$ (same row), or $j = j'$ (same column),
  or $\langle \lceil \frac{i}{3} \rceil, \lceil \frac{j}{3} \rceil \rangle = \langle \lceil \frac{i'}{3} \rceil, \lceil \frac{j'}{3} \rceil \rangle$ (same block)

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

D2.3 Assignments and Consistency

Assignments

Definition (assignment, partial assignment)

Let $\mathcal{C} = \langle V, \mathrm{dom}, (R_{uv}) \rangle$ be a constraint network.

A partial assignment of $\mathcal{C}$ (or of V) is a function

$$\alpha : V' \to \bigcup_{v \in V} \mathrm{dom}(v)$$

with $V' \subseteq V$ and $\alpha(v) \in \mathrm{dom}(v)$ for all $v \in V'$.

If $V' = V$, then α is also called total assignment (or assignment).

German: partielle Belegung, (totale) Belegung

- ⇝ partial assignments assign values to some or to all variables
- ⇝ (total) assignments are defined on all variables

Example

Partial Assignments of Running Example

$\alpha_1 = \{w \mapsto 1, z \mapsto 2\}$

$\alpha_2 = \{w \mapsto 3, x \mapsto 1\}$

Total Assignments of Running Example

$\alpha_3 = \{w \mapsto 1, x \mapsto 1, y \mapsto 2, z \mapsto 2\}$

$\alpha_4 = \{w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3\}$

Consistency

Definition (inconsistent, consistent, violated)

A partial assignment α of a constraint network $\mathcal{C}$ is called inconsistent if there are variables u, v such that α is defined for both u and v, and $\langle \alpha(u), \alpha(v) \rangle \notin R_{uv}$.

In this case, we say α violates the constraint R_{uv}.

A partial assignment is called consistent if it is not inconsistent.

German: inkonsistent, verletzt, konsistent

trivial example: The empty assignment is always consistent.

Example

Consistent Partial Assignment

$\alpha_1 = \{w \mapsto 1, z \mapsto 2\}$

Inconsistent Partial Assignment

$\alpha_2 = \{w \mapsto 2, x \mapsto 2\}$
violates $R_{wx} = \{\langle 2, 1\rangle, \langle 4, 2\rangle\}$

Inconsistent Assignment

$\alpha_3 = \{w \mapsto 2, x \mapsto 1, y \mapsto 2, z \mapsto 2\}$
violates $R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$ and
$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

Solution

Definition (solution, solvable)

Let $\mathcal{C}$ be a constraint network.

A consistent and total assignment of $\mathcal{C}$ is called a solution of $\mathcal{C}$.

If a solution of $\mathcal{C}$ exists, $\mathcal{C}$ is called solvable.

If no solution exists, $\mathcal{C}$ is called inconsistent.

German: Lösung, lösbar, inkonsistent

Solution of the Running Example

$\alpha = \{w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3\}$

Consistency vs. Solvability

Note: Consistent partial assignments α cannot necessarily be extended to a solution.

It only means that so far (i.e., on the variables where α is defined) no constraint is violated.

Example (4 queens problem): $\alpha = \{v_1 \mapsto 1, v_2 \mapsto 4, v_3 \mapsto 2\}$

	v_1	v_2	v_3	v_4
1	q			
2			q	
3				
4		q		

Complexity of Constraint Satisfaction Problems

Proposition (CSPs are NP-complete)

It is an NP-complete problem to decide whether a given constraint network is solvable.

Proof

Membership in NP:

Guess and check: guess a solution and check it for validity.
This can be done in polynomial time in the size of the input.

NP-hardness:

The graph coloring problem is a special case of CSPs
and is already known to be NP-complete.

Tightness of Constraint Networks

Definition (tighter, strictly tighter)

Let $\mathcal{C} = \langle V, \mathrm{dom}, R_{uv} \rangle$ and $\mathcal{C}' = \langle V, \mathrm{dom}', R'_{uv} \rangle$ be constraint networks with equal variable sets V.

$\mathcal{C}$ is called tighter than $\mathcal{C}'$, in symbols $\mathcal{C} \sqsubseteq \mathcal{C}'$, if

- $\mathrm{dom}(v) \subseteq \mathrm{dom}'(v)$ for all $v \in V$, and
- $R_{uv} \subseteq R'_{uv}$ for all $u, v \in V$ (including trivial constraints).

If at least one of these subset equations is strict, then $\mathcal{C}$ is called strictly tighter than $\mathcal{C}'$, in symbols $\mathcal{C} \sqsubset \mathcal{C}'$.

German: (echt) schärfer

Equivalence of Constraint Networks

Definition (equivalent)

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks with equal variable sets.

$\mathcal{C}$ and $\mathcal{C}'$ are called equivalent, in symbols $\mathcal{C} \equiv \mathcal{C}'$, if they have the same solutions.

German: äquivalent

D2.4 Outline and Summary

CSP Algorithms

In the following chapters, we will consider solution algorithms for constraint networks.

basic concepts:

- search: check partial assignments systematically
- backtracking: discard inconsistent partial assignments
- inference: derive equivalent, but tighter constraints to reduce the size of the search space

Summary

- formal definition of constraint networks:
  variables, domains, constraints
- compact encodings of exponentially many configurations
- unary and binary constraints
- assignments: partial and total
- consistency of assignments; solutions
- deciding solvability is NP-complete
- tightness of constraints
- equivalence of constraints