

# Foundations of Artificial Intelligence

## C1. Combinatorial Optimization: Introduction and Hill-Climbing

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April 2, 2025

# Combinatorial Optimization: Overview

Chapter overview: combinatorial optimization

- C1. Introduction and Hill-Climbing
- C2. Advanced Techniques

# Combinatorial Optimization

# Introduction

previous chapters: **classical state-space search**

- find **action sequence** (path) from initial to goal state
- difficulty: large number of states (**“state explosion”**)

next chapters: **combinatorial optimization**

↪ similar scenario, but:

- no actions or transitions
- don't search for path, but for **configuration** (“state”) with low cost/high quality

**German:** Zustandsraumexplosion, kombinatorische Optimierung, Konfiguration

# Combinatorial Optimization: Example

## Example: Nurse Scheduling Problem

- find a schedule for a hospital
- satisfy **hard constraints**
  - labor laws, hospital policies, . . .
  - nurses working night shifts should not work early next day
  - have enough nurses with required skills present at all times
- maximize satisfaction of **soft constraints**
  - individual preferences, reduce overtime, fair distribution, . . .

We are interested in a (high-quality) **schedule**, not a path to a goal.

# Combinatorial Optimization Problems

## Definition (combinatorial optimization problem)

A **combinatorial optimization problem** (COP) is given by a tuple  $\langle C, S, opt, v \rangle$  consisting of:

- a finite set of (solution) **candidates**  $C$
- a finite set of **solutions**  $S \subseteq C$
- an **objective sense**  $opt \in \{\min, \max\}$
- an **objective function**  $v : S \rightarrow \mathbb{R}$

**German:** kombinatorisches Optimierungsproblem, Kandidaten, Lösungen, Optimierungsrichtung, Zielfunktion

### Remarks:

- “problem” here in another sense (= “instance”) than commonly used in computer science
- practically interesting COPs usually have too many candidates to enumerate explicitly

# Optimal Solutions

## Definition (optimal)

Let  $\mathcal{O} = \langle C, S, opt, v \rangle$  be a COP.

The **optimal solution quality**  $v^*$  of  $\mathcal{O}$  is defined as

$$v^* = \begin{cases} \min_{c \in S} v(c) & \text{if } opt = \min \\ \max_{c \in S} v(c) & \text{if } opt = \max \end{cases}$$

( $v^*$  is undefined if  $S = \emptyset$ .)

A solution  $s$  of  $\mathcal{O}$  is called **optimal** if  $v(s) = v^*$ .

**German:** optimale Lösungsqualität, optimal

# Combinatorial Optimization

The basic algorithmic problem we want to solve:

## Combinatorial Optimization

Find a **solution** of good (ideally, optimal) quality for a combinatorial optimization problem  $\mathcal{O}$  or prove that no solution exists.

**Good** here means **close to  $v^*$**  (the closer, the better).



# Relevance and Hardness

- There is a huge number of practically important combinatorial optimization problems.
  - Solving these is a central focus of **operations research**.
  - Many important combinatorial optimization problems are **NP-complete**.
  - Most “classical” NP-complete problems can be formulated as combinatorial optimization problems.
- ↪ **Examples:** TSP, VERTEXCOVER, CLIQUE, BINPACKING, PARTITION

**German:** Unternehmensforschung, NP-vollständig

# Search vs. Optimization

Combinatorial optimization problems have

- a **search aspect** (among all candidates  $C$ , find a solution from the set  $S$ ) and
- an **optimization aspect** (among all solutions in  $S$ , find one of high quality).

# Pure Search/Optimization Problems

Important special cases arise when one of the two aspects is trivial:

- **pure search problems:**
  - all solutions are of equal quality
  - difficulty is in finding a solution **at all**
  - **formally:**  $v$  is a constant function (e.g., constant 0);  
 $opt$  can be chosen arbitrarily (does not matter)
- **pure optimization problems:**
  - all candidates are solutions
  - difficulty is in finding solutions of **high quality**
  - **formally:**  $S = C$

# Example

# Example: 8 Queens Problem

## 8 Queens Problem

How can we

- place **8 queens** on a chess board
- such that **no two queens threaten each other?**

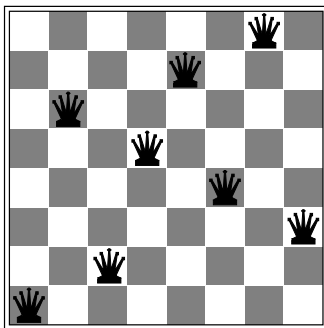
**German:** 8-Damen-Problem

- originally proposed in 1848
- **variants:** board size; other pieces; higher dimension

There are **92 solutions**, or **12 solutions** if we do not count symmetric solutions (under rotation or reflection) as distinct.

# Example: 8 Queens Problem

**Problem:** Place 8 queens on a chess board such that no two queens threaten each other.



Is this candidate a solution?

# Formally: 8 Queens Problem

How can we formalize the problem?

idea:

- obviously there must be exactly one queen in each file (“column”)
- describe candidates as 8-tuples, where the  $i$ -th entry denotes the rank (“row”) of the queen in the  $i$ -th file

formally:  $\mathcal{O} = \langle C, S, opt, v \rangle$  with

- $C = \{1, \dots, 8\}^8$
- $S = \{ \langle r_1, \dots, r_8 \rangle \mid \forall 1 \leq i < j \leq 8 : r_i \neq r_j \wedge |r_i - r_j| \neq |i - j| \}$
- $v$  constant,  $opt$  irrelevant (pure search problem)

# Local Search: Hill Climbing



# Algorithms for Combinatorial Optimization Problems

## How can we algorithmically solve COPs?

- formulation as classical state-space search
- formulation as constraint network
- formulation as logical satisfiability problem
- formulation as mathematical optimization problem (LP/IP)
- local search

# Algorithms for Combinatorial Optimization Problems

## How can we algorithmically solve COPs?

- formulation as classical state-space search  
  ↪ Part B
- formulation as constraint network ↪ Part D
- formulation as logical satisfiability problem ↪ Part E
- formulation as mathematical optimization problem (LP/IP)  
  ↪ not in this course
- **local search** ↪ today (Part C)

# Search Methods for Combinatorial Optimization

- main ideas of **heuristic search** applicable for COPs
  - ↪ states  $\approx$  candidates
- main difference: no “actions” in problem definition
  - instead, **we** (as algorithm designers) can choose which candidates to consider **neighbors**
  - definition of neighborhood **critical aspect** of designing good algorithms for a given COP
- “path to goal” irrelevant to the user
  - no path costs, parents or generating actions
  - ↪ no search nodes needed

# Local Search: Idea

## main ideas of local search algorithms for COPs:

- heuristic  $h$  estimates quality of candidates
  - for pure optimization: often objective function  $v$  itself
  - for pure search: often distance estimate to closest solution (as in state-space search)
- do not remember paths, only candidates
- often only **one** current candidate  $\rightsquigarrow$  very memory-efficient (however, not complete or optimal)
- often initialization with **random** candidate
- iterative improvement by **hill climbing**

# Hill Climbing

## Hill Climbing (for Maximization Problems)

*current* := a random candidate

**repeat:**

*next* := a neighbor of *current* with maximum *h* value

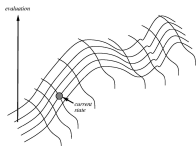
**if**  $h(\textit{next}) \leq h(\textit{current})$ :

**return** *current*

*current* := *next*

### Remarks:

- search as **walk** “uphill” in a **landscape** defined by the **neighborhood relation**
- heuristic values define “height” of terrain
- analogous algorithm for minimization problems also traditionally called “hill climbing” even though the metaphor does not fully fit



# Properties of Hill Climbing

- always terminates (**Why?**)
- no guarantee that result is a solution
- if result is a solution, it is **locally optimal** w.r.t.  $h$ , but no global quality guarantees

# Example: 8 Queens Problem

**Problem:** Place 8 queens on a chess board  
such that no two queens threaten each other.

**possible heuristic:** no. of pairs of queens threatening each other  
(formalization as minimization problem)

**possible neighborhood:** move one queen within its file

18	12	14	13	13	12	14	14
14	16	13	15	12	14	12	16
14	12	18	13	15	12	14	14
15	14	14	♚	13	16	13	16
♚	14	17	15	♚	14	16	16
17	♚	16	18	15	♚	15	♚
18	14	♚	15	15	14	♚	16
14	14	13	17	12	14	12	18

# Performance of Hill Climbing for 8 Queens Problem

- problem has  $8^8 \approx 17$  million candidates  
(reminder: 92 solutions among these)
- after random initialization, hill climbing finds a solution  
in around 14% of the cases
- only around 3–4 steps on average!



# Summary

# Summary

## combinatorial optimization problems:

- find **solution** of good **quality** (objective value) among many **candidates**
- special cases:
  - pure search problems
  - pure optimization problems
- differences to state-space search:  
no actions, paths etc.; only “state” matters

## often solved via **local search**:

- consider **one candidate** (or a few) at a time; try to improve it iteratively