# Foundations of Artificial Intelligence

## C1. Combinatorial Optimization: Introduction and Hill-Climbing

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# Combinatorial Optimization: Overview

Chapter overview: combinatorial optimization

- C1. Introduction and Hill-Climbing
- C2. Advanced Techniques

# C1.1 Combinatorial Optimization

# Introduction

previous chapters: classical state-space search

- find action sequence (path) from initial to goal state
- difficulty: large number of states ("state explosion")

next chapters: combinatorial optimization
$\leadsto$ similar scenario, but:

- no actions or transitions
- don't search for path, but for configuration ("state") with low cost/high quality

German: Zustandsraumexplosion, kombinatorische Optimierung, Konfiguration

# Combinatorial Optimization: Example

## Example: Nurse Scheduling Problem

- find a schedule for a hospital
- satisfy hard constraints
  - labor laws, hospital policies, ...
  - nurses working night shifts should not work early next day
  - have enough nurses with required skills present at all times
- maximize satisfaction of soft constraints
  - individual preferences, reduce overtime, fair distribution, ...

We are interested in a (high-quality) schedule, not a path to a goal.

# Combinatorial Optimization Problems

**Definition (combinatorial optimization problem)**

A combinatorial optimization problem (COP) is given by a tuple $\langle C, S, opt, v \rangle$ consisting of:

- a finite set of (solution) candidates $C$
- a finite set of solutions $S \subseteq C$
- an objective sense $opt \in \{\min, \max\}$
- an objective function $v : S \to \mathbb{R}$

German: kombinatorisches Optimierungsproblem, Kandidaten, Lösungen, Optimierungsrichtung, Zielfunktion

Remarks:

- "problem" here in another sense (= "instance") than commonly used in computer science
- practically interesting COPs usually have too many candidates to enumerate explicitly

# Optimal Solutions

**Definition (optimal)**

Let $\mathcal{O} = \langle C, S, opt, v \rangle$ be a COP.

The optimal solution quality $v^*$ of $\mathcal{O}$ is defined as

$$v^* = \begin{cases} \min_{c \in S} v(c) & \text{if } opt = \min \\ \max_{c \in S} v(c) & \text{if } opt = \max \end{cases}$$

($v^*$ is undefined if $S = \emptyset$.)

A solution $s$ of $\mathcal{O}$ is called optimal if $v(s) = v^*$.

German: optimale Lösungsqualität, optimal

# Combinatorial Optimization

The basic algorithmic problem we want to solve:

### Combinatorial Optimization

Find a solution of good (ideally, optimal) quality
for a combinatorial optimization problem $\mathcal{O}$
or prove that no solution exists.

Good here means close to $v^*$ (the closer, the better).

# Relevance and Hardness

- There is a huge number of practically important combinatorial optimization problems.
- Solving these is a central focus of operations research.
- Many important combinatorial optimization problems are NP-complete.
- Most "classical" NP-complete problems can be formulated as combinatorial optimization problems.

⇝ Examples: TSP, VertexCover, Clique, BinPacking, Partition

German: Unternehmensforschung, NP-vollständig

# Search vs. Optimization

Combinatorial optimization problems have

- a search aspect (among all candidates $C$, find a solution from the set $S$) and
- an optimization aspect (among all solutions in $S$, find one of high quality).

# Pure Search/Optimization Problems

Important special cases arise when one of the two aspects is trivial:

- pure search problems:
  - all solutions are of equal quality
  - difficulty is in finding a solution at all
  - formally: $v$ is a constant function (e.g., constant 0); *opt* can be chosen arbitrarily (does not matter)
- pure optimization problems:
  - all candidates are solutions
  - difficulty is in finding solutions of high quality
  - formally: $S = C$

# C1.2 Example

# Example: 8 Queens Problem

**8 Queens Problem**

How can we

- place 8 queens on a chess board
- such that no two queens threaten each other?

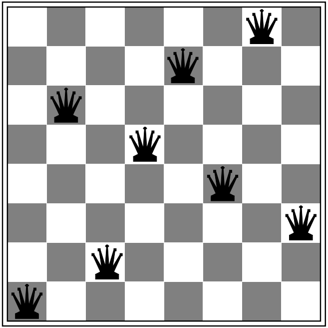
German: 8-Damen-Problem

- originally proposed in 1848
- variants: board size; other pieces; higher dimension

There are 92 solutions, or 12 solutions if we do not count symmetric solutions (under rotation or reflection) as distinct.

# Example: 8 Queens Problem

Problem: Place 8 queens on a chess board such that no two queens threaten each other.



Is this candidate a solution?

# Formally: 8 Queens Problem

How can we formalize the problem?

idea:

- obviously there must be exactly one queen in each file ("column")
- describe candidates as 8-tuples, where the $i$-th entry denotes the rank ("row") of the queen in the $i$-th file

formally: $\mathcal{O} = \langle C, S, opt, v \rangle$ with

- $C = \{1, \dots, 8\}^8$
- $S = \{\langle r_1, \dots, r_8 \rangle \mid \forall 1 \leq i < j \leq 8 : r_i \neq r_j \wedge |r_i - r_j| \neq |i - j|\}$
- $v$ constant, $opt$ irrelevant (pure search problem)

# C1.3 Local Search: Hill Climbing

# Algorithms for Combinatorial Optimization Problems

How can we algorithmically solve COPs?

- formulation as classical state-space search
  $\rightsquigarrow$ Part B
- formulation as constraint network $\rightsquigarrow$ Part D
- formulation as logical satisfiability problem $\rightsquigarrow$ Part E
- formulation as mathematical optimization problem (LP/IP)
  $\rightsquigarrow$ not in this course
- local search $\rightsquigarrow$ today (Part C)

# Search Methods for Combinatorial Optimization

- main ideas of heuristic search applicable for COPs
  $\rightsquigarrow$ states $\approx$ candidates
- main difference: no "actions" in problem definition
  - instead, we (as algorithm designers) can choose which candidates to consider neighbors
  - definition of neighborhood critical aspect of designing good algorithms for a given COP
- "path to goal" irrelevant to the user
  - no path costs, parents or generating actions
  - $\rightsquigarrow$ no search nodes needed

# Local Search: Idea

main ideas of local search algorithms for COPs:

- heuristic $h$ estimates quality of candidates
  - for pure optimization: often objective function $v$ itself
  - for pure search: often distance estimate to closest solution (as in state-space search)
- do not remember paths, only candidates
- often only one current candidate $\rightsquigarrow$ very memory-efficient (however, not complete or optimal)
- often initialization with random candidate
- iterative improvement by hill climbing

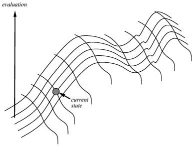
# Hill Climbing

**Hill Climbing (for Maximization Problems)**

*current* := a random candidate
**repeat**:
    *next* := a neighbor of *current* with maximum $h$ value
    **if** $h(next) \leq h(current)$:
        **return** *current*
    *current* := *next*

Remarks:

- search as walk "uphill" in a landscape defined by the neighborhood relation
- heuristic values define "height" of terrain
- analogous algorithm for minimization problems also traditionally called "hill climbing" even though the metaphor does not fully fit



# Properties of Hill Climbing

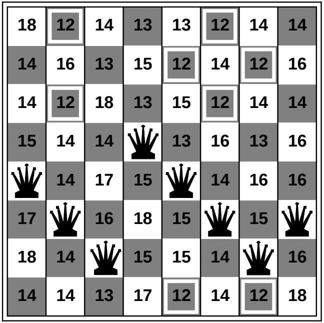
- always terminates (Why?)
- no guarantee that result is a solution
- if result is a solution, it is locally optimal w.r.t. $h$, but no global quality guarantees

# Example: 8 Queens Problem

**Problem:** Place 8 queens on a chess board such that no two queens threaten each other.

**possible heuristic:** no. of pairs of queens threatening each other (formalization as minimization problem)

**possible neighborhood:** move one queen within its file



# Performance of Hill Climbing for 8 Queens Problem

- problem has $8^8 \approx 17$ million candidates
  (reminder: 92 solutions among these)
- after random initialization, hill climbing finds a solution in around 14% of the cases
- only around 3–4 steps on average!

# C1.4 Summary

# Summary

combinatorial optimization problems:

- find solution of good quality (objective value) among many candidates
- special cases:
  - pure search problems
  - pure optimization problems
- differences to state-space search: no actions, paths etc.; only "state" matters

often solved via local search:

- consider one candidate (or a few) at a time; try to improve it iteratively