Foundations of Artificial Intelligence

B15. State-Space Search: Properties of A*, Part II

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State-Space Search: Overview

Chapter overview: state-space search

- B1–B3. Foundations
- B4–B8. Basic Algorithms
- B9–B15. Heuristic Algorithms
  - B9. Heuristics
  - B10. Analysis of Heuristics
  - B11. Best-first Graph Search
  - B12. Greedy Best-first Search, A^*, Weighted A^*
  - B13. IDA*
  - B14. Properties of A^*, Part I
  - B15. Properties of A^*, Part II

Introduction

Optimality of A* without Reopening

We now study A* without reopening.

- For A* without reopening, admissibility and consistency together guarantee optimality.
- We prove this on the following slides, again beginning with a basic lemma.
- Either of the two properties on its own would not be sufficient for optimality. (How would one prove this?)

Reminder: A* without Reopening

reminder from Chapter B11/B12: A* without reopening

A* without Reopening

```
open := new MinHeap ordered by ⟨f, h⟩
if h(init()) < ∞:
    open.insert(make_root_node())
closed := new HashSet
while not open.is_empty():
    n := open.pop_min()
    if n.state ∉ closed:
        closed.insert(n)
        if is_goal(n.state):
            return extract_path(n)
        for each ⟨a, s′⟩ ∈ succ(n.state):
            if h(s′) < ∞:
                n′ := make_node(n, a, s′)
                open.insert(n′)
return unsolvable
```

Monotonicity Lemma

A*: Monotonicity Lemma (1)

Lemma (monotonicity of A* with consistent heuristics)

Consider A* with a consistent heuristic.

Then:

1. *If n' is a child node of n, then $f(n') \geq f(n)$.*
2. *On all paths generated by A*, f values are non-decreasing.*
3. *The sequence of f values of the nodes expanded by A* is non-decreasing.*

German: Monotonielemma

A*: Monotonicity Lemma (2)

Proof.

on 1.:

Let n' be a child node of n via action a.

Let $s = n.\text{state}$, $s' = n'.\text{state}$.

A*: Monotonicity Lemma (2)

Proof.

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Let n' be a child node of n via action a.

Let $s = n.\text{state}$, $s' = n'.\text{state}$.

- by definition of f: $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$

A*: Monotonicity Lemma (2)

Proof.

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- by definition of f: $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$
- by definition of g: $g(n') = g(n) + cost(a)$

A*: Monotonicity Lemma (2)

Proof.

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Let n' be a child node of n via action a.

Let $s = n$.state, $s' = n'$.state.

- by definition of f: $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$
- by definition of g: $g(n') = g(n) + cost(a)$
- by consistency of h: $h(s) \leq cost(a) + h(s')$

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$\rightsquigarrow$ $f(n) = g(n) + h(s) \leq g(n) + cost(a) + h(s')$
$= g(n') + h(s') = f(n')$

A*: Monotonicity Lemma (2)

Proof.

on 1.:

Let n' be a child node of n via action a.

Let $s = n.\text{state}$, $s' = n'.\text{state}$.

- by definition of f: $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$
- by definition of g: $g(n') = g(n) + cost(a)$
- by consistency of h: $h(s) \leq cost(a) + h(s')$

$\rightsquigarrow$ $f(n) = g(n) + h(s) \leq g(n) + cost(a) + h(s')$
$= g(n') + h(s') = f(n')$

on 2.: follows directly from 1. ...

A*: Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*. Let n be the removed node with $f(n) = f_\mathrm{b}$.

A*: Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*. Let n be the removed node with $f(n) = f_\mathrm{b}$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b.

A*: Monotonicity Lemma (3)

Proof (continued).

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- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*. Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b.
- We must consider the operations modifying *open*: *open*.pop_min and *open*.insert.

A*: Monotonicity Lemma (3)

Proof (continued).

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- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*. Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b.
- We must consider the operations modifying *open*: *open*.pop_min and *open*.insert.
- *open*.pop_min can never decrease the minimal f value in *open* (only potentially increase it).

A*: Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*. Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b.
- We must consider the operations modifying *open*: *open*.pop_min and *open*.insert.
- *open*.pop_min can never decrease the minimal f value in *open* (only potentially increase it).
- The nodes n' added with *open*.insert are children of n and hence satisfy $f(n') \geq f(n) = f_b$ according to part 1.

□

Optimality of A* without Reopening

Optimality of A* without Reopening

Theorem (optimality of A* without reopening)

A* without reopening is optimal when using an admissible and consistent heuristic.

Proof.

From the monotonicity lemma, the sequence of f values of nodes removed from the open list is non-decreasing.

- $\rightsquigarrow$ If multiple nodes with the same state s are removed from the open list, then their g values are non-decreasing.
- $\rightsquigarrow$ If we allowed reopening, it would never happen.
- $\rightsquigarrow$ With consistent heuristics, A* without reopening behaves the same way as A* with reopening.

The result follows because A* with reopening and admissible heuristics is optimal. □

Time Complexity of A*

Time Complexity of A* (1)

What is the time complexity of A*?

- depends strongly on the quality of the heuristic
- an extreme case: $h = 0$ for all states
  - $\rightsquigarrow$ A* identical to uniform cost search
- another extreme case: $h = h^*$ and $cost(a) > 0$ for all actions a
  - $\rightsquigarrow$ A* only expands nodes along an optimal solution
  - $\rightsquigarrow$ $O(\ell^*)$ expanded nodes, $O(\ell^* b)$ generated nodes, where
    - ℓ^*: length of the found optimal solution
    - b: branching factor

Time Complexity of A* (2)

more precise analysis:

- dependency of the runtime of A* on heuristic error

example:

- unit cost problems with
- constant branching factor and
- constant absolute error: $|h^*(s) - h(s)| \leq c$ for all $s \in S$

time complexity:

- if state space is a tree: time complexity of A* grows linearly in solution length (Pohl 1969; Gaschnig 1977)
- general search spaces: runtime of A* grows exponentially in solution length (Helmert & Röger 2008)

Overhead of Reopening

How does reopening affect runtime?

- For most practical state spaces and inconsistent admissible heuristics, the number of reopened nodes is negligible.
- exceptions exist:
  Martelli (1977) constructed state spaces with n states
  where exponentially many (in n) node reopenings occur in A^*.
  ($\rightsquigarrow$ exponentially worse than uniform cost search)

Practical Evaluation of A* (1)

9	2	12	6
5	7	14	13
3		1	11
15	4	10	8

$\longrightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

h_1: number of tiles in wrong cell (misplaced tiles)

h_2: sum of distances of tiles to their goal cell (Manhattan distance)

Practical Evaluation of A* (2)

- experiments with random initial states, generated by random walk from goal state
- entries show median of number of generated nodes for 101 random walks of the same length N

	generated nodes		
N	BFS-Graph	A* with h_1	A* with h_2
10	63	15	15
20	1,052	28	27
30	7,546	77	42
40	72,768	227	64
50	359,298	422	83
60	$>$ 1,000,000	7,100	307
70	$>$ 1,000,000	12,769	377
80	$>$ 1,000,000	62,583	849
90	$>$ 1,000,000	162,035	1,522
100	$>$ 1,000,000	690,497	4,964

Summary

Summary

- A^* without reopening using an admissible and consistent heuristic is optimal
- key property monotonicity lemma (with consistent heuristics):
  - f values never decrease along paths considered by A^*
  - sequence of f values of expanded nodes is non-decreasing
- time complexity depends on heuristic and shape of state space
  - precise details complex and depend on many aspects
  - reopening increases runtime exponentially in degenerate cases, but usually negligible overhead
  - small improvements in heuristic values often lead to exponential improvements in runtime