Foundations of Artificial Intelligence

B14. State-Space Search: Properties of A*, Part I

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State-Space Search: Overview

Chapter overview: state-space search

- B1–B3. Foundations
- B4–B8. Basic Algorithms
- B9–B15. Heuristic Algorithms
  - B9. Heuristics
  - B10. Analysis of Heuristics
  - B11. Best-first Graph Search
  - B12. Greedy Best-first Search, A*, Weighted A*
  - B13. IDA*
  - B14. Properties of A*, Part I
  - B15. Properties of A*, Part II

Introduction

Optimality of A*

- advantage of A* over greedy search:
  optimal for heuristics with suitable properties
- very important result!

⇝ next chapters: a closer look at A*

- A* with reopening ⇝ this chapter
- A* without reopening ⇝ next chapter

Optimality of A* with Reopening

In this chapter, we prove that A* with reopening is optimal when using admissible heuristics.

For this purpose, we

- give some basic definitions
- prove two lemmas regarding the behaviour of A*
- use these to prove the main result

Reminder: A* with Reopening

reminder from Chapter B11/B12: A* with reopening

A* with Reopening

```
open := new MinHeap ordered by ⟨f, h⟩
if h(init()) < ∞:
    open.insert(make_root_node())
distances := new HashMap
while not open.is_empty():
    n := open.pop_min()
    if distances.lookup(n.state) = none or g(n) < distances[n.state]:
        distances[n.state] := g(n)
        if is_goal(n.state):
            return extract_path(n)
        for each ⟨a, s'⟩ ∈ succ(n.state):
            if h(s') < ∞:
                n' := make_node(n, a, s')
                open.insert(n')
return unsolvable
```

Solvable States

Definition (solvable)

A state s of a state space is called solvable if $h^*(s) < \infty$.

German: lösbar

Optimal Paths to States

Definition (g^*)

Let s be a state of a state space with initial state s_I.

We write $g^*(s)$ for the cost of an optimal (cheapest) path from s_I to s (∞ if s is unreachable).

Remarks:

- g is defined for nodes, g^* for states (Why?)
- $g^*(n.\text{state}) \leq g(n)$ for all nodes n generated by a search algorithm (Why?)

Settled States in A*

Definition (settled)

A state s is called settled at a given point during the execution of A* (with or without reopening) if s is included in *distances* and $distances[s] = g^*(s)$.

German: erledigt

Optimal Continuation Lemma

Optimal Continuation Lemma

We now show the first important result for A* with reopening:

Lemma (optimal continuation lemma)

Consider A* with reopening using a safe heuristic at the beginning of any iteration of the **while** *loop.*

If

- *state s is settled,*
- *state s' is a solvable successor of s, and*
- *an optimal path from s_I to s' of the form $\langle s_I, \ldots, s, s' \rangle$ exists,*

then

- *s' is settled or*
- *open contains a node n' with $n'.\mathrm{state} = s'$ and $g(n') = g^*(s')$.*

German: Optimale-Fortsetzungs-Lemma

Optimal Continuation Lemma: Intuition

(Proof follows on the next slides.)

Intuitively, the lemma states:

> *If no optimal path to a given state has been found yet, open must contain a "good" node that contributes to finding an optimal path to that state.*

(This potentially requires multiple applications of the lemma along an optimal path to the state.)

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration ("iteration A") of A*.

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Because s is settled, an earlier iteration ("iteration B") set $distances[s] := g^*(s)$.

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration ("iteration A") of A*.

Because s is settled, an earlier iteration ("iteration B") set $distances[s] := g^*(s)$.

Thus iteration B removed a node n with $n.\mathrm{state} = s$ and $g(n) = g^*(s)$ from *open*.

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration ("iteration A") of A*.

Because s is settled, an earlier iteration ("iteration B") set $distances[s] := g^*(s)$.

Thus iteration B removed a node n with n.state $= s$ and $g(n) = g^*(s)$ from *open*.

A* did not terminate in iteration B.
(Otherwise iteration A would not exist.)
Hence n was expanded in iteration B. ...

Optimal Continuation Lemma: Proof (2)

Proof (continued).

This expansion considered the successor s' of s.
Because s' is solvable, we have $h^*(s') < \infty$.
Because h is safe, this implies $h(s') < \infty$.
Hence a successor node n' was generated for s'.

Optimal Continuation Lemma: Proof (2)

Proof (continued).

This expansion considered the successor s' of s.
Because s' is solvable, we have $h^*(s') < \infty$.
Because h is safe, this implies $h(s') < \infty$.
Hence a successor node n' was generated for s'.

This node n' satisfies the consequence of the lemma.
Hence the criteria of the lemma were satisfied for s and s' after iteration B.

Optimal Continuation Lemma: Proof (2)

Proof (continued).

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Because s' is solvable, we have $h^*(s') < \infty$.
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Hence a successor node n' was generated for s'.

This node n' satisfies the consequence of the lemma.
Hence the criteria of the lemma were satisfied for s and s' after iteration B.

To complete the proof, we show: if the consequence of the lemma is satisfied at the beginning of an iteration, it is also satisfied at the beginning of the next iteration. ...

Optimal Continuation Lemma: Proof (3)

Proof (continued).

- If s' is settled at the beginning of an iteration, it remains settled until termination.

Optimal Continuation Lemma: Proof (3)

Proof (continued).

- If s' is settled at the beginning of an iteration, it remains settled until termination.
- If s' is not yet settled and *open* contains a node n' with $n'.\text{state} = s'$ and $g(n') = g^*(s')$ at the beginning of an iteration, then either the node remains in *open* during the iteration, or n' is removed during the iteration and s' becomes settled.

□

f-Bound Lemma

f-Bound Lemma

We need a second lemma:

Lemma (f-bound lemma)

Consider A* with reopening and an admissible heuristic applied to a solvable state space with optimal solution cost c^*.

Then open contains a node n with $f(n) \leq c^*$ at the beginning of each iteration of the **while** *loop.*

German: f-Schranken-Lemma

f-Bound Lemma: Proof (1)

Proof.

Consider the situation at the beginning of any iteration of the **while** loop.

Let $\langle s_0, \dots, s_n \rangle$ with $s_0 := s_I$ be an optimal solution.
(Here we use that the state space is solvable.)

f-Bound Lemma: Proof (1)

Proof.

Consider the situation at the beginning of any iteration of the **while** loop.

Let $\langle s_0, \dots, s_n \rangle$ with $s_0 := s_\mathrm{I}$ be an optimal solution. (Here we use that the state space is solvable.)

Let s_i be the first state in the sequence that is not settled.

(Not all states in the sequence can be settled: s_n is a goal state, and when a goal state is inserted into *distances*, A* terminates.) ...

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because $s_0 = s_\mathrm{I}$ is not settled yet, we are at the first iteration of the **while** loop.

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because $s_0 = s_I$ is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and h is admissible, we have $h(s_0) < \infty$.

f-Bound Lemma: Proof (2)

Proof (continued).

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Because $s_0 = s_\mathrm{I}$ is not settled yet, we are at the first iteration of the **while** loop.

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Hence *open* contains the root n_0.

f-Bound Lemma: Proof (2)

Proof (continued).

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Because $s_0 = s_\text{I}$ is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and h is admissible, we have $h(s_0) < \infty$.

Hence *open* contains the root n_0.

We obtain: $f(n_0) = g(n_0) + h(s_0) = 0 + h(s_0) \leq h^*(s_0) = c^*$, where "$\leq$" uses the admissibility of h.

This concludes the proof for this case. . . .

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.
Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i.

f-Bound Lemma: Proof (3)

Proof (continued).

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We can hence apply the optimal continuation lemma (with $s = s_{i-1}$ and $s' = s_i$) and obtain:

(A) s_i is settled, or

(B) *open* contains n' with $n'.\text{state} = s_i$ and $g(n') = g^*(s_i)$.

f-Bound Lemma: Proof (3)

Proof (continued).

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Because (A) is false, (B) must be true.

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.
Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i.

We can hence apply the optimal continuation lemma (with $s = s_{i-1}$ and $s' = s_i$) and obtain:

(A) s_i is settled, or

(B) *open* contains n' with $n'.\text{state} = s_i$ and $g(n') = g^*(s_i)$.

Because (A) is false, (B) must be true.

We conclude: open contains n' with
$f(n') = g(n') + h(s_i) = g^*(s_i) + h(s_i) \leq g^*(s_i) + h^*(s_i) = c^*$,
where "$\leq$" uses the admissibility of h. □

Optimality of A* with Reopening

Optimality of A* with Reopening

We can now show the main result of this chapter:

Theorem (optimality of A* with reopening)

A with reopening is optimal when using an admissible heuristic.*

Optimality of A* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

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This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

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With $h(n.\text{state}) = 0$ (because h is admissible and hence goal-aware), this implies:

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This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

With $h(n.\text{state}) = 0$ (because h is admissible and hence goal-aware), this implies:

$$f(n) = g(n) + h(n.\text{state}) = g(n) + 0 = g(n) = c > c^*.$$

Optimality of A* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

With $h(n.\text{state}) = 0$ (because h is admissible and hence goal-aware), this implies:

$$f(n) = g(n) + h(n.\text{state}) = g(n) + 0 = g(n) = c > c^*.$$

A* always removes a node n with minimal f value from *open*. With $f(n) > c^*$, we get a contradiction to the f-bound lemma, which completes the proof. □

Summary

Summary

- A* with reopening using an admissible heuristic is optimal.
- The proof is based on the following lemmas that hold for solvable state spaces and admissible heuristics:
  - optimal continuation lemma: The open list always contains nodes that make progress towards an optimal solution.
  - f-bound lemma: The minimum f value in the open list at the beginning of each A* iteration is a lower bound on the optimal solution cost.