Foundations of Artificial Intelligence

B6. State-Space Search: Breadth-first Search

Malte Helmert

University of Basel

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Chapter overview: state-space search

- B1–B3. Foundations
- B4–B8. Basic Algorithms
  - B4. Data Structures for Search Algorithms
  - B5. Tree Search and Graph Search
  - B6. Breadth-first Search
  - B7. Uniform Cost Search
  - B8. Depth-first Search and Iterative Deepening
- B9–B15. Heuristic Algorithms

Blind Search

In Chapters B6–B8 we consider blind search algorithms:

Blind Search Algorithms

Blind search algorithms use no information
about state spaces apart from the black box interface.

They are also called uninformed search algorithms.

contrast: heuristic search algorithms (Chapters B9–B15)

Blind Search Algorithms: Examples

examples of blind search algorithms:

- breadth-first search
- uniform cost search
- depth-first search
- depth-limited search
- iterative deepening search

Blind Search Algorithms: Examples

examples of blind search algorithms:

- breadth-first search ($\rightsquigarrow$ this chapter)
- uniform cost search
- depth-first search
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Blind Search Algorithms: Examples

examples of blind search algorithms:

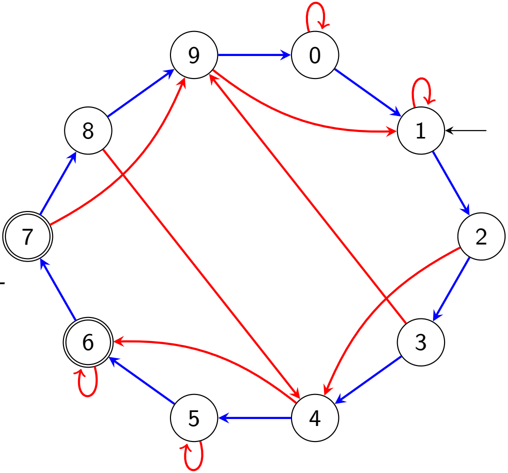
- breadth-first search ($\rightsquigarrow$ this chapter)
- uniform cost search ($\rightsquigarrow$ Chapter B7)
- depth-first search ($\rightsquigarrow$ Chapter B8)
- depth-limited search ($\rightsquigarrow$ Chapter B8)
- iterative deepening search ($\rightsquigarrow$ Chapter B8)

Breadth-first Search: Introduction

Running Example: Reminder

bounded inc-and-square:

- $S = \{0, 1, \dots, 9\}$
- $A = \{inc, sqr\}$
- $cost(inc) = cost(sqr) = 1$
- T s.t. for $i = 0, \dots, 9$:
  - $\langle i, inc, (i+1) \bmod 10 \rangle \in T$
  - $\langle i, sqr, i^2 \bmod 10 \rangle \in T$
- $s_\mathrm{I} = 1$
- $S_\mathrm{G} = \{6, 7\}$



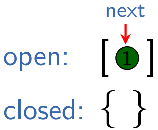
Idea

breadth-first search:

- expand nodes in order of generation (FIFO)
  - $\rightsquigarrow$ open list is linked list or deque
- we start with an example using graph search

German: Breitensuche

Example: Generic Graph Search with FIFO Expansion



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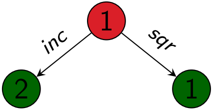
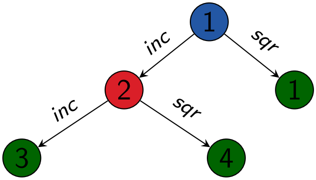


next

open: [2 1]

closed: $\{1\}$

Example: Generic Graph Search with FIFO Expansion

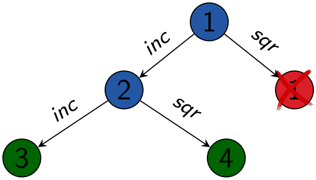


next

open: [1 3 4]

closed: $\{1, 2\}$

Example: Generic Graph Search with FIFO Expansion



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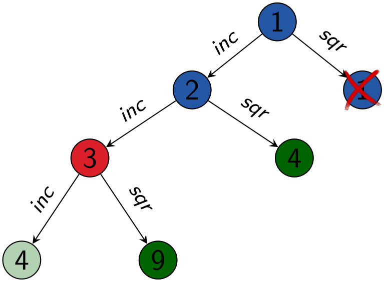
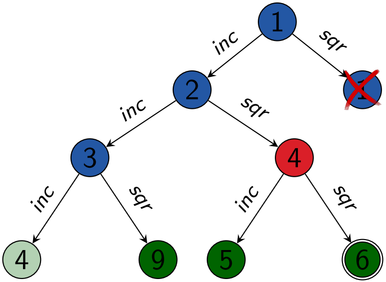


next

open: [4 4 9]

closed: $\{1, 2, 3\}$

Example: Generic Graph Search with FIFO Expansion

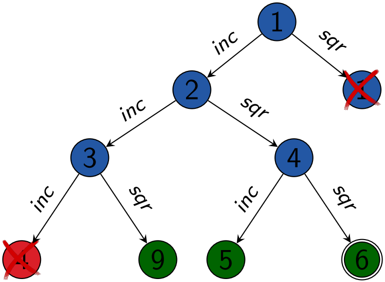


next

open: [4 9 5 6]

closed: $\{1, 2, 3, 4\}$

Example: Generic Graph Search with FIFO Expansion

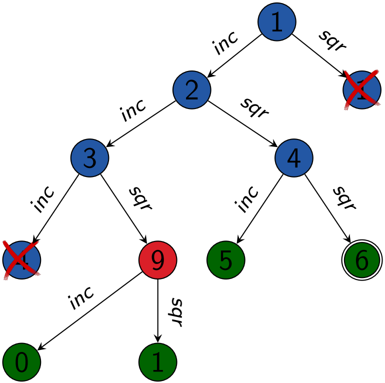


next

open: [9 5 6]

closed: $\{1, 2, 3, 4\}$

Example: Generic Graph Search with FIFO Expansion

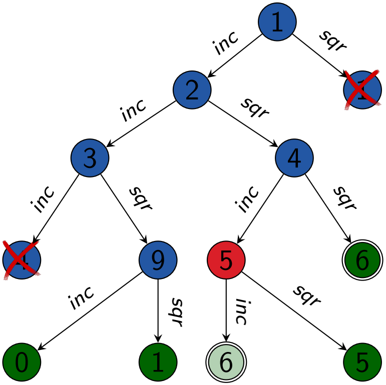


next

open: [5 6 0 1]

closed: $\{1, 2, 3, 4, 9\}$

Example: Generic Graph Search with FIFO Expansion

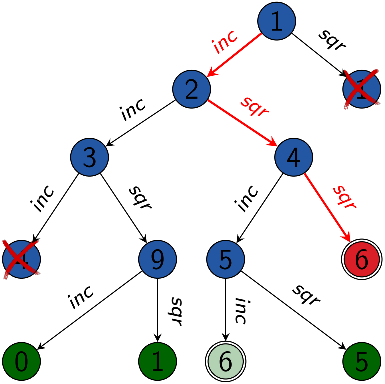


next

open: $[\ 6\ 0\ 1\ 6\ 5\ ]$

closed: $\{1, 2, 3, 4, 5, 9\}$

Example: Generic Graph Search with FIFO Expansion



next

open: [0 1 6 5]

closed: $\{1, 2, 3, 4, 5, 6, 9\}$

Observations from Example

breadth-first search behaviour:

- state space is searched layer by layer

$\rightsquigarrow$ shallowest goal node is always found first

Breadth-first Search: Tree Search or Graph Search?

Breadth-first search can be performed

- without duplicate elimination (as a tree search)
  $\rightsquigarrow$ BFS-Tree
- or with duplicate elimination (as a graph search)
  $\rightsquigarrow$ BFS-Graph

(BFS = breadth-first search).

$\rightsquigarrow$ We consider both variants.

BFS-Tree

Reminder: Generic Tree Search Algorithm

reminder from Chapter B5:

Generic Tree Search

$open := \textbf{new}$ OpenList
$open$.insert(make_root_node())
while not $open$.is_empty():
    $n := open$.pop()
    if is_goal(n.state):
        return extract_path(n)
    for each $\langle a, s' \rangle \in$ succ(n.state):
        $n' :=$ make_node(n, a, s')
        $open$.insert(n')
return unsolvable

BFS-Tree (1st Attempt)

breadth-first search without duplicate elimination (1st attempt):

BFS-Tree (1st Attempt)

open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    if is_goal(n.state):
        return extract_path(n)
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
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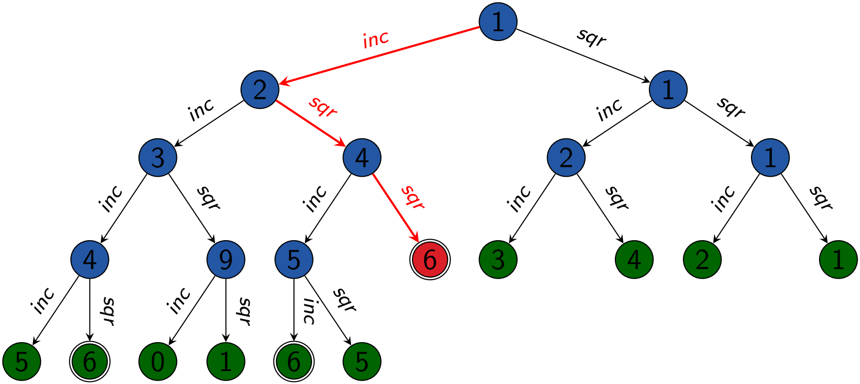
        n' := make_node(n, a, s')
        open.push_back(n')
return unsolvable

Running Example: BFS-Tree (1st Attempt)



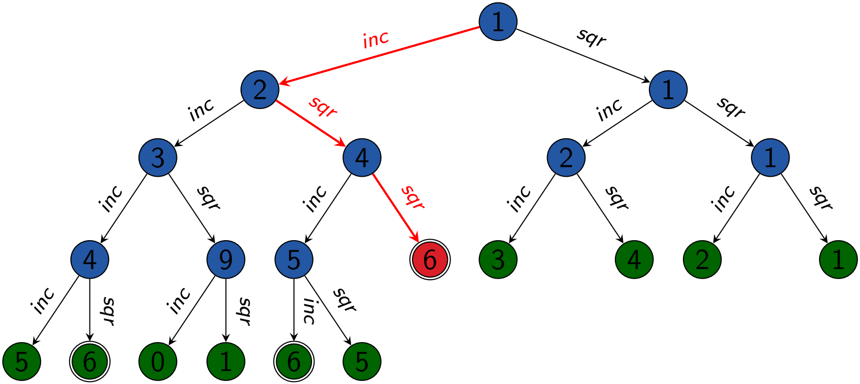
Opportunities for Improvement

- In a BFS, the first generated goal node is always the first expanded goal node. (Why?)

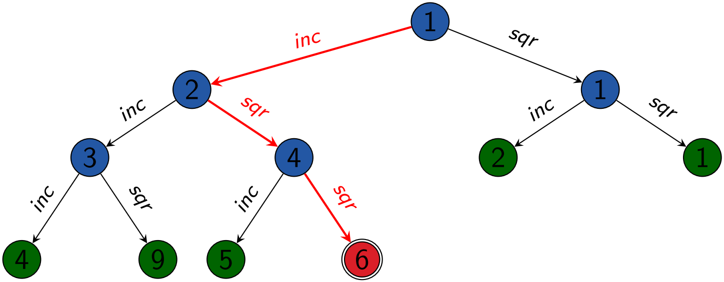
$\leadsto$ It is more efficient to perform the goal test upon generating a node (rather than upon expanding it).

$\leadsto$ How much effort does this save?

BFS-Tree without Early Goal Tests



BFS-Tree with Early Goal Tests



BFS-Tree (2nd Attempt)

breadth-first search without duplicate elimination (2nd attempt):

BFS-Tree (2nd Attempt)

open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    ~~**if** is_goal(n.state):~~
        ~~**return** extract_path(n)~~
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        open.push_back(n')
return unsolvable

BFS-Tree (2nd Attempt)

breadth-first search without duplicate elimination (2nd attempt):

BFS-Tree (2nd Attempt)

open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    ~~**if** is_goal(*n*.state):~~
        ~~**return** extract_path(*n*)~~
    for each $\langle a, s' \rangle \in$ succ(*n*.state):
        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        open.push_back(n')
return unsolvable

BFS-Tree (2nd Attempt): Discussion

Where is the bug?

BFS-Tree (Final Version)

breadth-first search without duplicate elimination (final version):

BFS-Tree

if is_goal(init()):
    return $\langle\rangle$
open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        open.push_back(n')
return unsolvable

BFS-Tree (Final Version)

breadth-first search without duplicate elimination (final version):

BFS-Tree

if is_goal(init()):
  return $\langle\rangle$
open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
  n := *open*.pop_front()
  for each $\langle a, s'\rangle \in$ succ(n.state):
    n' := make_node(n, a, s')
    if is_goal(s'):
      return extract_path(n')
    open.push_back(n')
return unsolvable

BFS-Graph

Reminder: Generic Graph Search Algorithm

reminder from Chapter B5:

Generic Graph Search

```
open := new OpenList
open.insert(make_root_node())
closed := new ClosedList
while not open.is_empty():
    n := open.pop()
    if closed.lookup(n.state) = none:
        closed.insert(n)
        if is_goal(n.state):
            return extract_path(n)
        for each ⟨a, s'⟩ ∈ succ(n.state):
            n' := make_node(n, a, s')
            open.insert(n')
return unsolvable
```

Adapting Generic Graph Search to Breadth-First Search

Adapting the generic algorithm to breadth-first search:

- similar adaptations to BFS-Tree
  (deque as open list, early goal tests)
- as closed list does not need to manage node information, a set data structure suffices
- for the same reasons why early goal tests are a good idea, we should perform duplicate tests against the closed list and updates of the closed lists as early as possible

BFS-Graph (Breadth-First Search with Duplicate Elim.)

BFS-Graph

```
if is_goal(init()):
    return ⟨⟩
open := new Deque
open.push_back(make_root_node())
closed := new HashSet
closed.insert(init())
while not open.is_empty():
    n := open.pop_front()
    for each ⟨a, s′⟩ ∈ succ(n.state):
        n′ := make_node(n, a, s′)
        if is_goal(s′):
            return extract_path(n′)
        if s′ ∉ closed:
            closed.insert(s′)
            open.push_back(n′)
return unsolvable
```

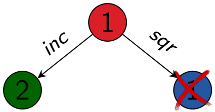
BFS-Graph: Example

next

open: $[\,1\,]$

closed: $\{1\}$

BFS-Graph: Example

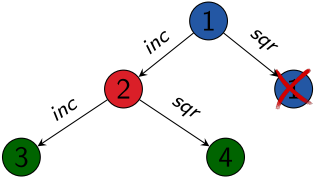


next

open: [2]

closed: $\{1,2\}$

BFS-Graph: Example

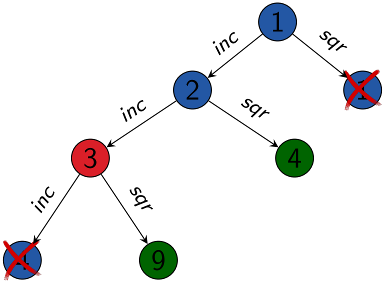


next

open: [3 4]

closed: $\{1, 2, 3, 4\}$

BFS-Graph: Example

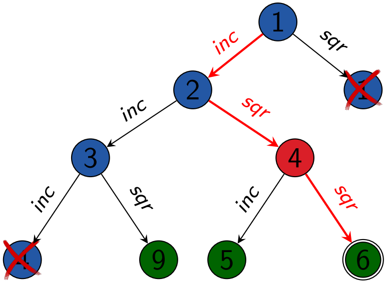


next

open: [4 9]

closed: $\{1, 2, 3, 4, 9\}$

BFS-Graph: Example



next

open: [9 5]

closed: $\{1, 2, 3, 4, 5, 9\}$

Properties of Breadth-first Search

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Properties of Breadth-first Search:

- BFS-Tree is semi-complete, but not complete. (Why?)
- BFS-Graph is complete. (Why?)
- BFS (both variants) is optimal
  if all actions have the same cost (Why?),
  but not in general (Why not?).
- complexity: next slides

Breadth-first Search: Complexity

The following result applies to both BFS variants:

Theorem (time complexity of breadth-first search)

Let b be the branching factor and d be the minimal solution length of the given state space. Let $b \geq 2$.

Then the time complexity of breadth-first search is

$$1 + b + b^2 + b^3 + \cdots + b^d = O(b^d)$$

Reminder: we measure time complexity in generated nodes.

It follows that the space complexity of both BFS variants also is $O(b^d)$ (if $b \geq 2$). (Why?)

Breadth-first Search: Example of Complexity

example: $b = 13$; 100 000 nodes/second; 32 bytes/node

d	nodes	time	memory
4	30 940	0.3 s	966 KiB
6	$5.2 \cdot 10^6$	52 s	159 MiB
8	$8.8 \cdot 10^8$	147 min	26 GiB
10	10^{11}	17 days	4.3 TiB
12	10^{13}	8 years	734 TiB
14	10^{15}	1 352 years	121 PiB
16	10^{17}	$2.2 \cdot 10^5$ years	20 EiB
18	10^{20}	$38 \cdot 10^6$ years	3.3 ZiB

Breadth-first Search: Example of Complexity

example: $b = 13$; 100 000 nodes/second; 32 bytes/node

Realistic numbers?

d	nodes	time	memory
4	30 940	0.3 s	966 KiB
6	$5.2 \cdot 10^6$	52 s	159 MiB
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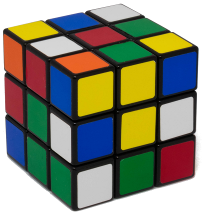

Rubik's cube:

- branching factor: ≈ 13
- typical solution length: 18

d	nodes	time	memory
4	30 940	0.3 s	966 KiB
6	$5.2 \cdot 10^6$	52 s	159 MiB
8	$8.8 \cdot 10^8$	147 min	26 GiB
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BFS-Tree or BFS-Graph?

Which is better, BFS-Tree or BFS-Graph?

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advantages of BFS-Graph:

- complete
- much (!) more efficient if there are many duplicates

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advantages of BFS-Tree:

- simpler
- less overhead (time/space) if there are few duplicates

BFS-Tree or BFS-Graph?

Which is better, BFS-Tree or BFS-Graph?

advantages of BFS-Graph:

- complete
- much (!) more efficient if there are many duplicates

advantages of BFS-Tree:

- simpler
- less overhead (time/space) if there are few duplicates

Conclusion

BFS-Graph is usually preferable, unless we know that there is a negligible number of duplicates in the given state space.

Summary

Summary

- blind search algorithm: use no information except black box interface of state space
- breadth-first search: expand nodes in order of generation
  - search state space layer by layer
  - can be tree search or graph search
  - complexity $O(b^d)$ with branching factor b, minimal solution length d (if $b \geq 2$)
  - complete as a graph search; semi-complete as a tree search
  - optimal with uniform action costs