

Algorithms and Data Structures

C5. Shortest Paths: Foundations

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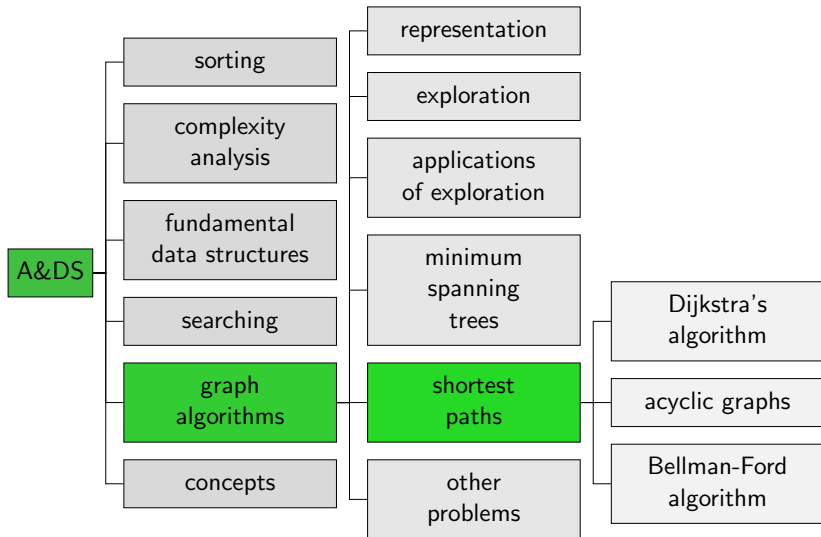
C5.1 Introduction

C5.2 Foundations

C5.3 Optimality Criterion and Generic Algorithm

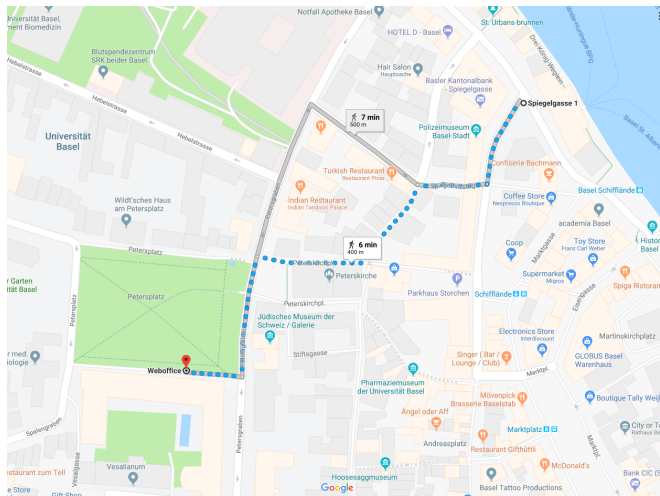
C5.4 Summary

Content of the Course



C5.1 Introduction

Google Maps



Seam Carving



Applications

- ▶ Route planning
- ▶ Path planning in games
- ▶ robot navigation
- ▶ seam carving
- ▶ automated planning
- ▶ typesetting in TeX
- ▶ routing protocols in networks (OSPF, BGP, RIP)
- ▶ routing of telecommunication messages
- ▶ traffic routing

Source (partially): Network Flows: Theory, Algorithms, and Applications,
R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993

Variants

What are we interested in?

- ▶ **Single source:** from one vertex s to all other vertices
- ▶ **Single sink:** from all vertices to one vertex t
- ▶ **Source-sink:** from vertex s to vertex t
- ▶ **All pairs:** from every vertex to every vertex

Graph properties

- ▶ arbitrary / non-negative / Euclidean weights
- ▶ arbitrary / non-negative / no cycles

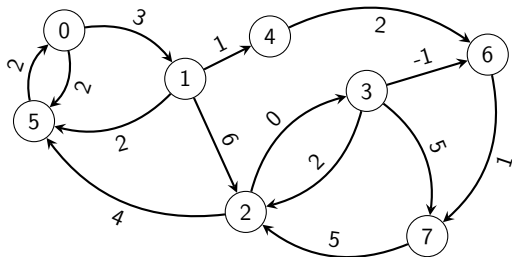
C5.2 Foundations

Weighted Directed Graphs

Same (high-level) definition of weighted graphs as before, but now we consider directed graphs.

Directed Graph

An (edge)-weighted graph associates every edge e with a weight (or cost) $weight(e) \in \mathbb{R}$.



Reminder: A directed graphs is also called a **digraph**.

API for Weighted Directed Edge

```
1  class DirectedEdge:
2      # Edge from n1 to n2 with weight w
3      def __init__(n1: int, n2: int, w: float) -> None
4
5      # Weight of the edge
6      def weight() -> float
7
8      # Initial vertex of the edge
9      def from_node() -> int
10
11     # Terminal vertex of the edge
12     def to_node() -> int
```

API for Weighted Digraphs

```
1  class EdgeWeightedDigraph:
2      # Graph with no_nodes vertices and no edges
3      def __init__(no_nodes: int) -> None
4
5      # Add weighted edge
6      def add_edge(e: DirectedEdge) -> None
7
8      # Number of vertices
9      def no_nodes() -> int
10
11     # Number of edges
12     def no_edges() -> int
13
14     # All outgoing edges of n
15     def outgoing_edges(n: int) -> Generator[DirectedEdge]
16
17     # All edges
18     def all_edges() -> Generator[DirectedEdge]
```

Shortest Path Problem

Single-source shortest path problem, SSSP

- ▶ Given: Graph and start vertex s
- ▶ Query for vertex v
 - ▶ Is there a path from s to v ?
 - ▶ If yes, what is the shortest path?
- ▶ In **weighted graphs**:
Shortest path is the one with **lowest weight**
(= minimal sum of edge costs)

API for Shortest-path Implementation

The algorithms for shortest paths should implement the following interface:

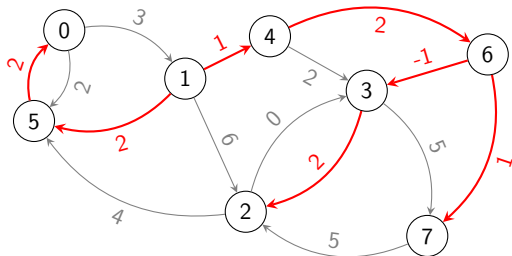
```
1 class ShortestPaths:
2     # Initialization for start vertex s
3     def __init__(graph: EdgeWeightedDigraph, s: int) -> None
4
5     # Distance from s to v; infinity, if there is no path
6     def dist_to(v: int) -> float
7
8     # Is there a path from s to v?
9     def has_path_to(v: int) -> bool
10
11    # Path from s to v; None, if there is none
12    def path_to(v: int) -> Generator[DirectedEdge]
```

Shortest-path Tree

Shortest-path Tree

For a weighted digraph G and vertex s , a **shortest-path tree** is a subgraph that

- ▶ forms a directed tree with root s ,
- ▶ contains all vertices that are reachable from s , and
- ▶ for which every path in the tree is a shortest path in G .



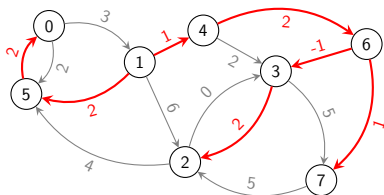
Shortest-path Tree: Representation

Representation: arrays indexed by vertex

- ▶ **parent** with reference to parent vertex
None for unreachable vertices and start vertex
- ▶ **distance** with distance from the start vertex
 ∞ for unreachable vertices

	0	1	2	3	4	5	6	7
parent	5	None	3	6	1	1	4	6

	0	1	2	3	4	5	6	7
distance	4	0	4	2	1	2	3	4



What about parallel edges?

Extracting Shortest Paths

```
1  def path_to(self, node):
2      if self.distance[node] == float('inf'):
3          yield None
4      elif self.parent[node] is None:
5          yield node
6      else:
7          # output path from start to parent node
8          self.path_to(self.parent[node])
9          # finish with node
10         yield node
```

This implementation generates a sequence of vertices. What do we have to change to generate a corresponding sequence of edges?

Relaxation

Relaxing edge (u, v)

- ▶ $\text{distance}[u]$: cost of the shortest **known** path to u
- ▶ $\text{distance}[v]$: cost of the shortest **known** path to v
- ▶ $\text{parent}[v]$: predecessor of v in the shortest known paths to v
- ▶ Does edge (u, v) establish a shorter path to v (through u)?
- ▶ If yes, update $\text{distance}[v]$ and $\text{parent}[v]$.

Illustration: Whiteboard

Relaxation

```
1  def relax(self, edge):
2      u = edge.from_node()
3      v = edge.to_node()
4      if self.distance[v] > self.distance[u] + edge.weight():
5          self.parent[v] = u
6          self.distance[v] = self.distance[u] + edge.weight()
```

C5.3 Optimality Criterion and Generic Algorithm

Optimality Criterion

Theorem

Let G be a weighted digraph without negative cycles.

Array $distance[]$ contains the cost of the shortest paths from s if and only if

- ❶ $distance[s] = 0$
- ❷ $distance[w] \leq distance[v] + weight(e)$
for all edges $e = (v, w)$, and
- ❸ *for all vertices v , $distance[v]$ is the cost of **some** path from s to v , or ∞ if there is no such path.*

Optimality Criterion (Continued)

Proof

“ \Rightarrow ”

- ① Since the graph has no cycles of negative cost, no path from s to s can have negative cost. Thus, the empty path is optimal and $\text{distance}[s]$ is 0.
- ② Consider an arbitrary edge e from u to v .
The shortest path from s to u has cost $\text{distance}[u]$. If we extend this path by edge e , we have a path from s to v of cost $\text{distance}[u] + \text{weight}(e)$. Thus, the cost of a shortest path from s to v cannot be larger and it holds that $\text{distance}[v] \leq \text{distance}[u] + \text{weight}(e)$.
- ③ Trivially true.

...

Optimality Criterion (Continued)

Proof (continued).

“ \Leftarrow ”

For unreachable vertices, the value is infinity by definition.

Consider an arbitrary vertex v and a shortest path $p = (v_0, \dots, v_n)$ from s to v , i.e. $v_0 = s$, $v_n = v$.

For $i \in \{1, \dots, n\}$, let e_i be a cheapest edge from v_{i-1} to v_i .

Since all inequalities are satisfied, we have

$$\begin{aligned} \text{distance}[v_n] &\leq \text{distance}[v_{n-1}] + \text{weight}(e_n) \\ &\leq \text{distance}[v_{n-2}] + \text{weight}(e_{n-1}) + \text{weight}(e_n) \\ &\leq \dots \leq \text{weight}(e_1) + \dots + \text{weight}(e_n) \\ &= \text{cost of an optimal path.} \end{aligned}$$

Due to 3, $\text{distance}[v_n]$ cannot be lower than the optimal cost. \square

Generic Algorithm

Generic Algorithm for Start Vertex s

- ▶ Initialize $\text{distance}[s] = 0$ and $\text{distance}[v] = \infty$ for all other vertices
- ▶ As long as the optimality criterion is not satisfied:
Relax an arbitrary edge

Correct:

- ▶ Finite $\text{distance}[v]$ always corresponds to the cost of a path from s to v .
- ▶ Every successful relaxation reduces $\text{distance}[v]$ for some v .
- ▶ For every vertex, the distance can only be reduced finitely often.

C5.4 Summary

Summary

- ▶ **Single-source shortest paths:** Compute in a weighted digraph the shortest paths from a given vertex to all reachable vertices.
- ▶ **Relaxation:** If for edge (u,v) the best known distance to v is larger than the one to u plus the edge cost, then update the distance to v (with predecessor u).
- ▶ **Generic algorithm**
 - ▶ Based on relaxation and optimality criterion.
 - ▶ Every instantiation is correct for all weighted digraphs **without negative-cost cycles**.
 - ▶ Specific instantiations: next chapter.