Algorithms and Data Structures

C3. Disjoint-set Data Structure/Union-Find

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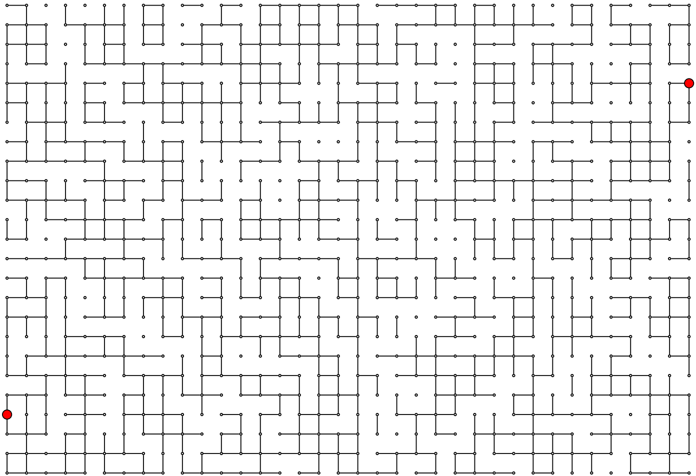
Union-Find

Questions



Are the red vertices connected?

How many connected components does the graph have?

Connected Components as Disjoint Sets

Set of conn. components as collection of disjoint sets of objects.

- One set for all vertices of one connected component.
- Operations:
  - Union: Given two objects, merge the sets that contain them into one.
    Introduce a new edge between the given vertices, connecting their connected components.
  - Find: Given an object, return a representative of the set that contains it.
    Given a vertex, return a representative vertex for its connected component.
    - Must return the same representative for all objects in the set.
    - The representative may only change if set gets merged.
    - Two objects are in the same set (two vertices are connected) if find returns the same representative for them.
  - Count: Return the number of sets
    Return the number of connected components.

Union-Find Data Type

```
class UnionFind:
  # Initialization for n objects (with names 0, ..., n-1).
  def __init__(n: int) -> None

  # Merge the sets containing objects v and w.
  def union(v: int, w: int) -> None

  # Representative for set containing v.
  # May change if set is merged by call of union,
  # but not otherwise.
  def find(v: int) -> int

  # Number of sets.
  def count() -> int
```

(Somewhat) Naive Algorithm: Quick-Find

- For n objects: Array `representative` of length n.
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- Entry at position i is representative of the set containing i.
- Initially, every object is (alone) in its own set, and thus its representative.
- Update the array in every call of `union`.

Quick-Find Data Structure

```
class QuickFind:
    def __init__(self, no_nodes):
        self.components = no_nodes
        self.representative = list(range(no_nodes))

    def count(self):
        return self.components

    def find(self, v):
        return self.representative[v]
```

[0, 1, ..., no_nodes-1]

Quick-Find Data Structure (Continued)

```python
    def union(self, v, w):
        repr_v = self.find(v)
        repr_w = self.find(w)
        if repr_v == repr_w:  # already in same component
            return
        # replace all occurrences of repr_v in
        # self.representative with repr_w
        for i in range(len(self.representative)):
            if self.representative[i] == repr_v:
                self.representative[i] = repr_w
        self.components -= 1  # we merged two components
```

Running time?

- Cost model = number of array accesses
- one access for every call of `find`
- between   and   accesses
  for every call of `union` that merges two components

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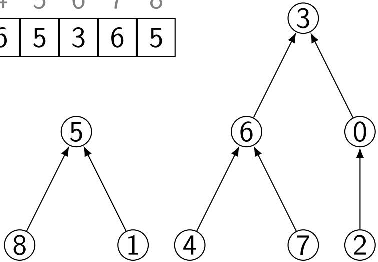
Running time?

- Cost model = number of array accesses
- one access for every call of `find`
- between $n + 3$ and $2n + 1$ accesses
  for every call of `union` that merges two components

Better: Quick-Union aka Disjoint-set Forest

- (implicit) tree for representing each set
- represented as array with parent nodes as entries (root: reference to itself)

0	1	2	3	4	5	6	7	8
3	5	0	3	6	5	3	6	5



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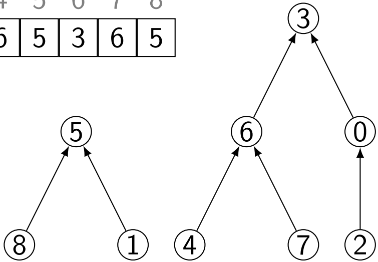

- Root node serves as representative of the set.

Quick-Union Data Structure

```python
class QuickUnion:
    def __init__(self, no_nodes):
        self.parent = list(range(no_nodes))
        self.components = no_nodes

    def find(self, v):
        while self.parent[v] != v:
            v = self.parent[v]
        return v

    def union(self, v, w):
        repr_v = self.find(v)
        repr_w = self.find(w)
        if repr_v == repr_w:  # already in same component
            return
        self.parent[repr_v] = repr_w
        self.components -= 1

    # count as in QuickFind
```

First Improvement

- Problem with Quick-Union: Trees can degenerate into chains.
  → `find` requires linear time in the size of the set.
- Idea: In union the root of the tree with lower height becomes a child of the root of the higher tree.

Ranked Quick-Union Algorithm

```
class RankedQuickUnion:
    def __init__(self, no_nodes):
        self.parent = list(range(no_nodes))
        self.components = no_nodes
        self.rank = [0] * no_nodes  # [0, ..., 0]

    def union(self, v, w):
        repr_v = self.find(v)
        repr_w = self.find(w)
        if repr_v == repr_w:
            return
        if self.rank[repr_w] < self.rank[repr_v]:
            self.parent[repr_w] = repr_v
        else:
            self.parent[repr_v] = repr_w
            if self.rank[repr_v] == self.rank[repr_w]:
                self.rank[repr_w] += 1
        self.components -= 1

    # connected, count and find as in QuickUnion
```

Second Improvement

Path Compression

- Idea: During find, reconnect all traversed nodes to the root.
- We do not update the height of the tree during path compression.
  - Value of `rank` can deviate from the actual height.
  - That's why it is called rank and not height.

Ranked Quick-Union Algorithm with Path Compression

```
class RankedQuickUnionWithPathCompression:
    def __init__(self, no_nodes):
        self.parent = list(range(no_nodes))
        self.components = no_nodes
        self.rank = [0] * no_nodes  # [0, ..., 0]

    def find(self, v):
        if self.parent[v] == v:
            return v
        root = self.find(self.parent[v])
        self.parent[v] = root
        return root

    # connected, count and union as in RankedQuickUnion
```

Discussion

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  - m calls of `find` for n objects (and at most $n - 1$ calls of `union`, merging two components)
  - $O(m\alpha(m, n))$ array accesses
  - α is inverse of a variant of the Ackermann function
  - In practise is $\alpha(m, n) \leq 3$.

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  - $O(m\alpha(m, n))$ array accesses
  - α is inverse of a variant of the Ackermann function
  - In practise is $\alpha(m, n) \leq 3$.
- Nevertheless: there cannot be a union-find structure that guarantees linear running time.

  (under cell-probe model, only accounting for memory access)

Comparison to Exploration-based Approach

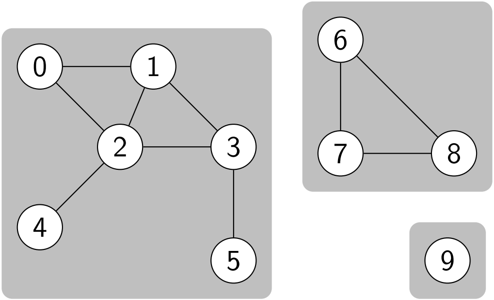
- Chapter C2: Algorithm `ConnectedComponents`, based on graph exploration.
- After the precomputation, queries only require constant time.
- In practise, disjoint-set forests are often faster, because for many applications, we do not have to build up the full graph.
- If the graph has already been built up, graph exploration can be better.
- Another advantage of union find:
  - Online approach
  - We can easily introduce further edges.

Connected Components and Equivalence Classes

Reminder: Connected Components

Undirected graph

- Two vertices u and v are in the same connected component if there is a path between u and v (= vertices u and v are connected).



Connected Components: Properties

- The connected components define a partition of the vertices:
  - Every vertex is in a connected component.
  - No vertex is in more than one connected component.
- "is connected with" is an equivalence relation.
  - reflexive: Every vertex is connected with itself.
  - symmetric: If u is connected with v, then v is connected with u.
  - transitive: If u is connected with v, and v with w, then u is connected with w.

Partition in General

Definition (Partition)

A partition of a finite set M is a set P of non-empty subsets of M, such that

- every element of M is in some set in P:
  $\bigcup_{S \in P} S = M$, and
- that sets in P are pairwise disjoint:
  $S \cap S' = \emptyset$ for $S, S' \in P$ with $S \neq S'$.

The sets in P are called blocks.

$M = \{e_1, \ldots, e_5\}$

- $P_1 = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$
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Equivalence Relations in General

Definition (Equivalence Relation)

An equivalence relation over set M is a symmetric, transitive and reflexive relation $R \subseteq M \times M$.
We write $a \sim b$ for $(a, b) \in R$ and say that a is equivalent to b.

- symmetric: $a \sim b$ implies $b \sim a$
- transitive: $a \sim b$ and $b \sim c$ implies $a \sim c$
- reflexive: for all $e \in M$: $e \sim e$

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- Vice versa:
  For partition P define $R = \{(x, y) \mid \exists B \in P : x, y \in B\}$
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  (i.e. $x \sim y$ if and only if x and y are in the same block).
  Then R is an equivalence relation.
- We can consider blocks in partitions as equivalence classes and vice versa.

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  - no "unnecessary" equivalences.

Can use union-find data structures to determine equivalence classes.

Summary

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- A union-find data structure maintains a collection of disjoint sets.
  - union: merge two sets.
  - find: identify the set containing an object and return its representative.
- Good implementation: Disjoint-set forest
  with improvements to keep the height of the trees low:
  - Union adjoins the shorter tree to the taller tree.
  - Find reconnects traversed nodes to the root
    (path compression).
- Applications:
  - Connected components
  - Finest equivalence relation