Algorithms and Data Structures

B6. Red-Black Trees

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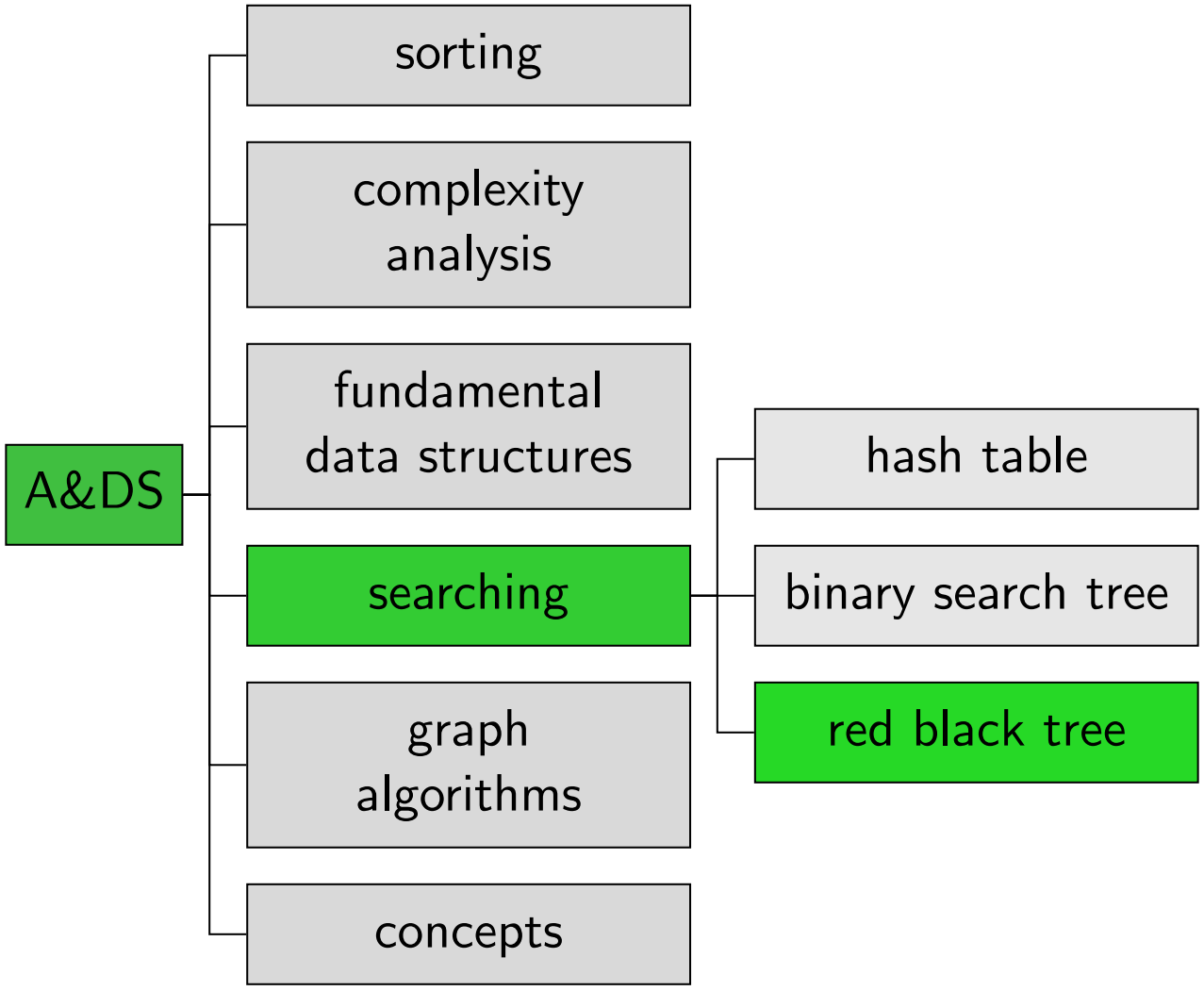
Red-Black Trees

Content of the Course



Motivation

- Binary search trees can support many relevant operations in linear time in the height of the tree.
- But: Binary search trees can degenerate into chains, in which case the operations take linear time in the number n of elements (no better than with a linked list).
- Idea: Search-trees schemes that are in some form "balanced" and can guarantee running time $O(\log_2 n)$ in the worst case.
  - AVL trees: for every node, the height of the left and right subtree differs by at most 1.
  - B-trees: permit several keys and subtrees per node (e.g. special case: 2-3 tree).
  - Red-Black trees: use node colors to maintain an approximate balancing.
  - ...

Red-Black Trees: Representation

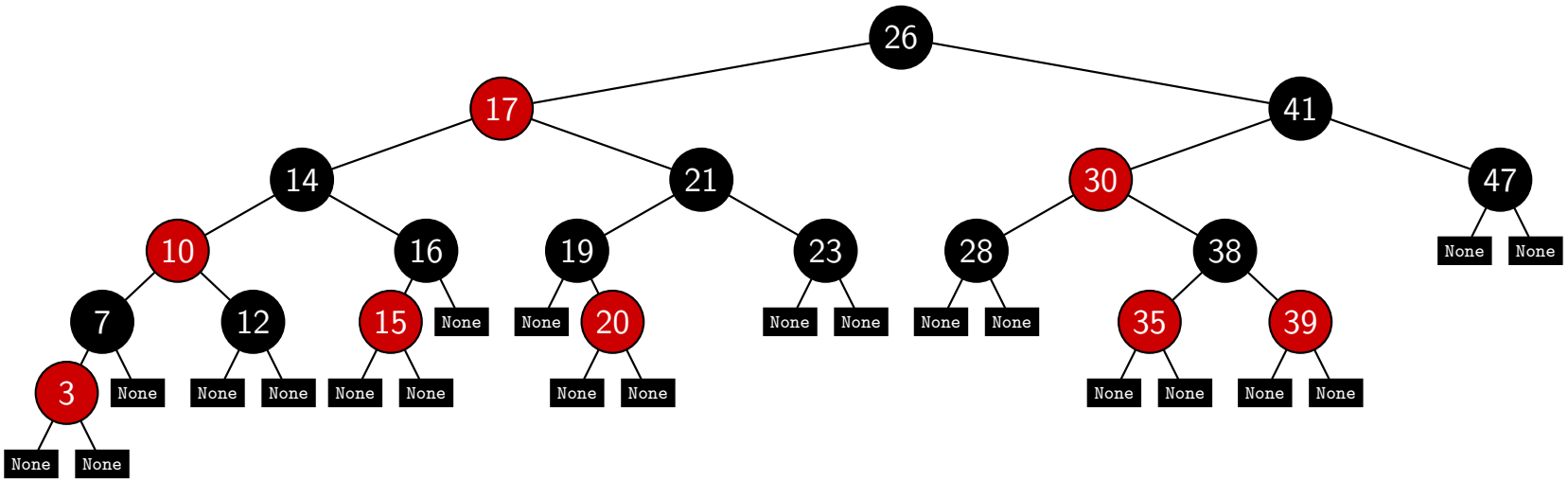
- Use one extra bit per node, storing its color, which can be either red or black.
- Each node now contains attributes color, key, left, right and parent.

`None` Leaf Nodes

- Left, right and parent are `None` if there is no corresponding node.

None Leaf Nodes

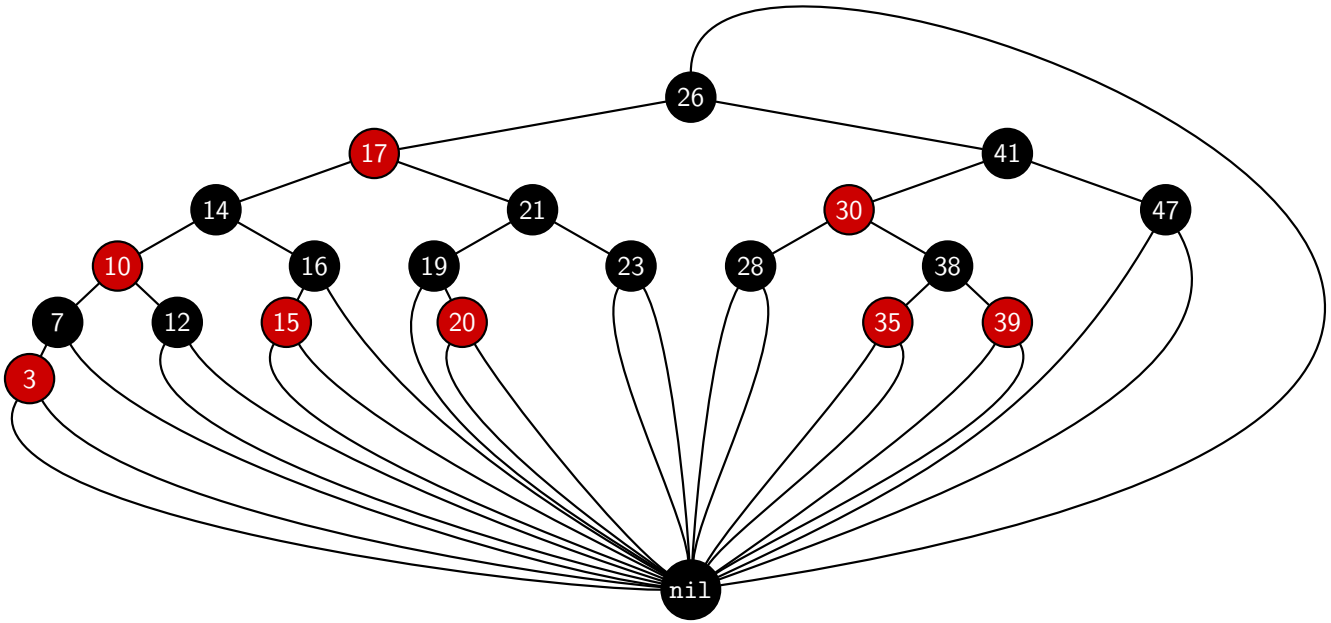
- Left, right and parent are `None` if there is no corresponding node.
- Because it is conceptionally and implementation-wise easier, we will represent them as actual node objects.
- These are then the leaves of the trees and the nodes holding the entries are inner nodes.



None Leaf Nodes: Sentinel

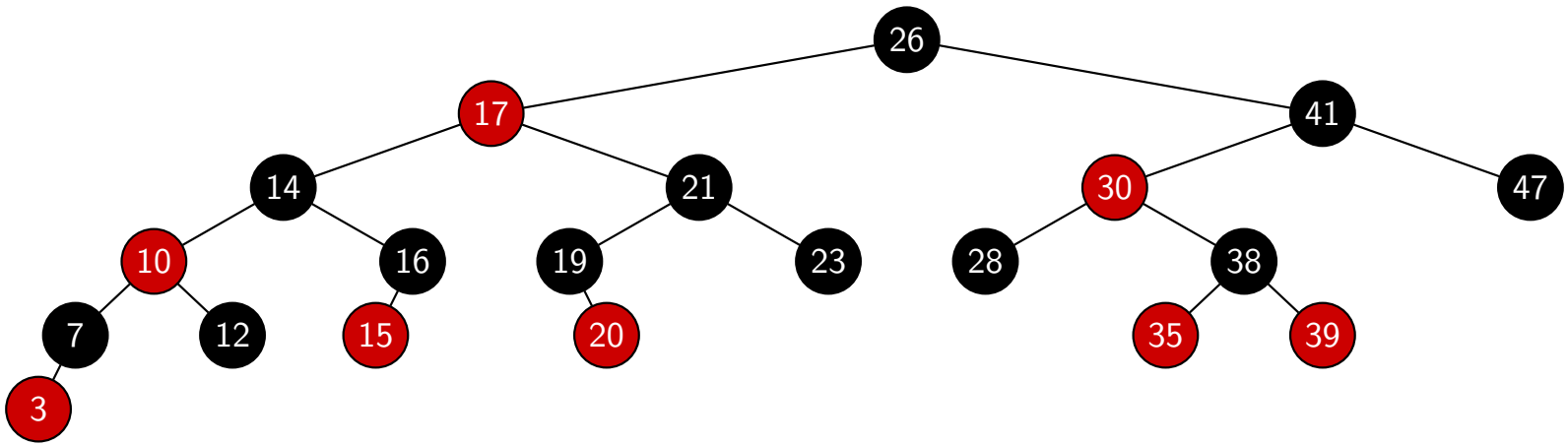
Instead of many leaf nodes, we use a single sentinel node `nil`.

- Implemented like a normal (black) node
  but used as child of many nodes.
- The sentinel also serves as parent of the root.
- Attributes for parent and children can take on arbitrary values.



Graphical Representation

On the slides, we omit the `None` leaf nodes/sentinel:



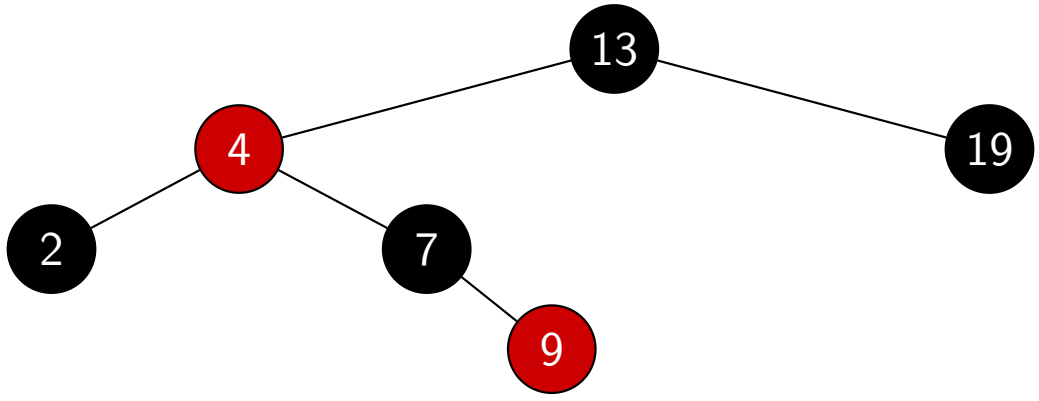
Red-Black Trees

Definition (Red-Black Tree)

A red-black tree is a binary search tree that satisfies the following red-black properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Quiz I: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

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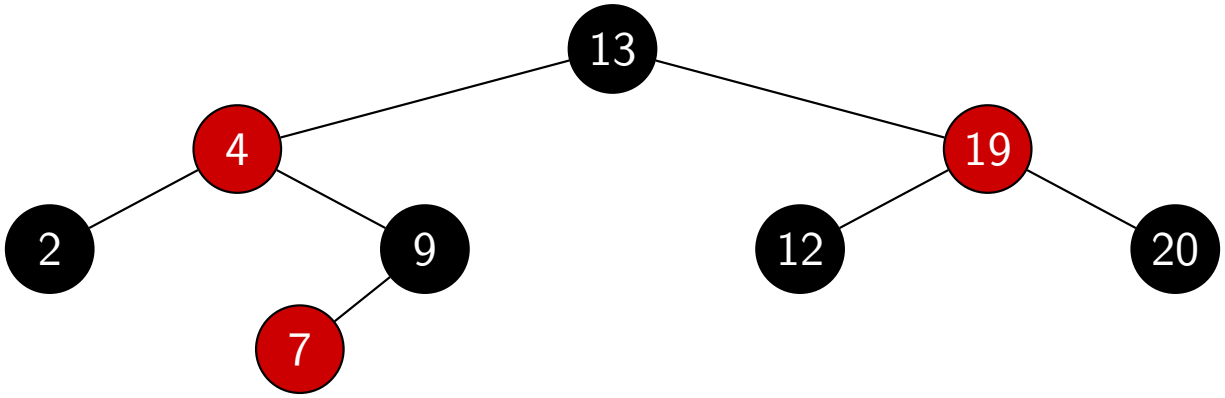

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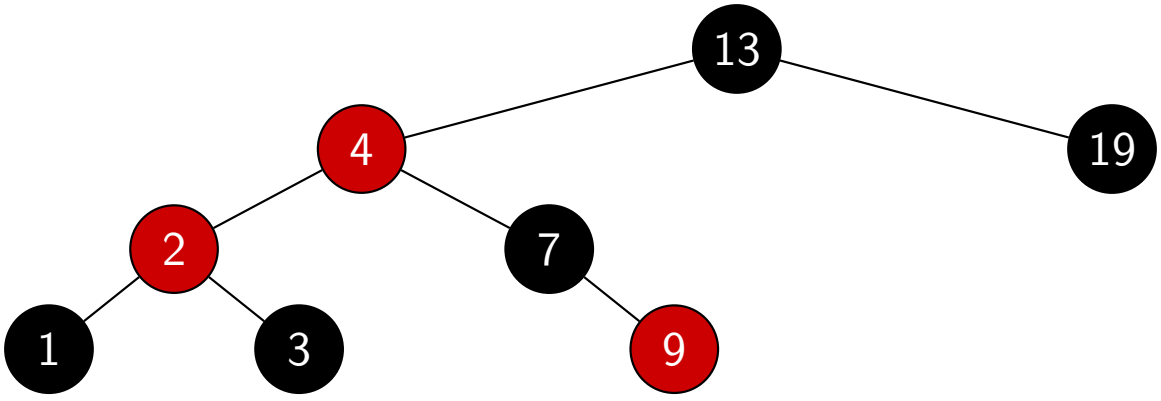

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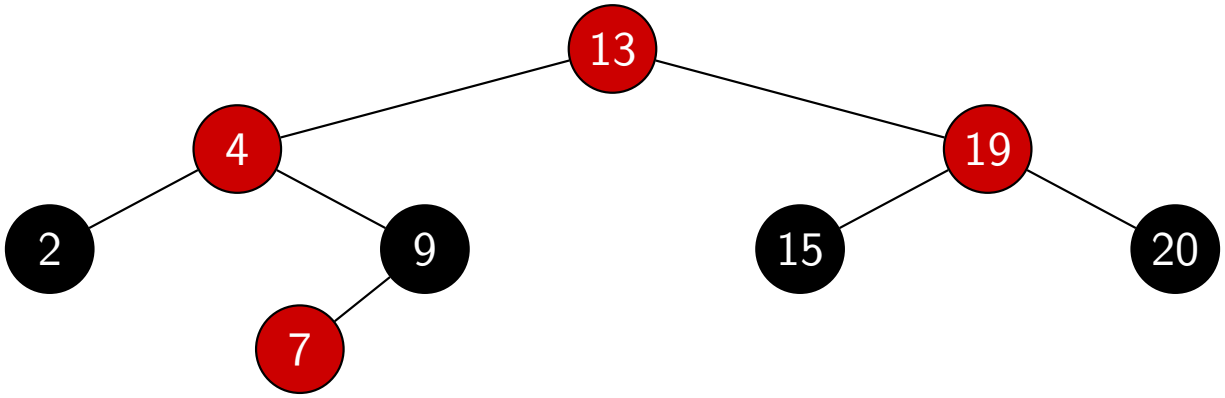

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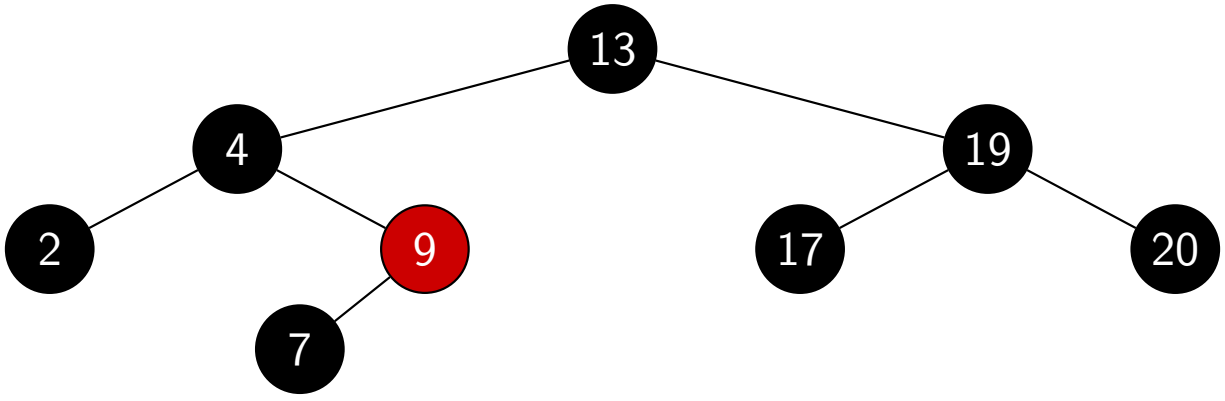

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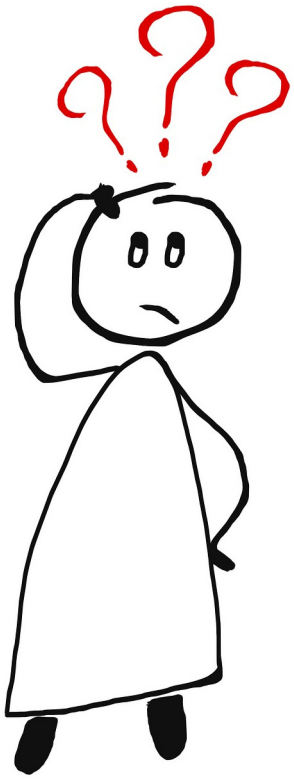
Questions



Questions?

Height of Red-Black Tree

Theorem

A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

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Let the black-height $\mathrm{bh}(x)$ of node x denote the number of black nodes on any simple path from, but not including, x down to a leaf.

We first show by induction on the height of x that the subtree rooted at any node x contains at least $2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

Height of Red-Black Tree

Proof (continued).

Height of x is 0: x is a leaf and the subtree rooted at x contains $2^{\mathrm{bh}(x)} - 1 = 2^0 - 1 = 0$ inner nodes.

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Inductive step: x has positive height.

Then x has two children. If a child is black, it contributes 1 to x's black-height but not to its own. If a child is red, then it contributes to neither x's black-height nor its own.

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Therefore, each child has a black-height of $\mathrm{bh}(x) - 1$ or $\mathrm{bh}(x)$.

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Since the height of the child is smaller than the one of x, by the inductive hypothesis the subtree rooted by each child has at least $2^{\mathrm{bh}(x)-1} - 1$ inner nodes.

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Thus, the subtree rooted by x contains at least $2(2^{\mathrm{bh}(x)-1} - 1) + 1 = 2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

Height of Red-Black Tree

Proof (continued).

We showed that that the subtree rooted at any node x contains at least $2^{\mathrm{bh}(x)} - 1$ inner nodes.

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Thus, the black-height of the root is at least $h/2$ and thus $n > 2^{h/2} - 1$.

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Moving the 1 to the left-hand side and taking logarithms on both sides yields $\log_2(n+1) \geq h/2$, or $h \leq 2\log_2(n+1)$. □

Height of Red-Black Tree: Consequence

Theorem

A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

- The height of a red-black tree is in $O(\log_2 n)$.
- Red-black trees are binary search trees.
- On binary search trees, `search(n, k)`, `minimum(n)`, `maximum(n)`, `successor(n)`,`predecessor(n)` can run in time $O(h)$ (cf. Ch. B5).
- We can use the same implementation for red-black trees, achieving running time $O(\log_2(n))$ for all these queries.

Insertion (and Deletion)

Modifying Red-Black Trees

We cannot simply use the insertion and deletion implementation from binary search trees (Why not?).

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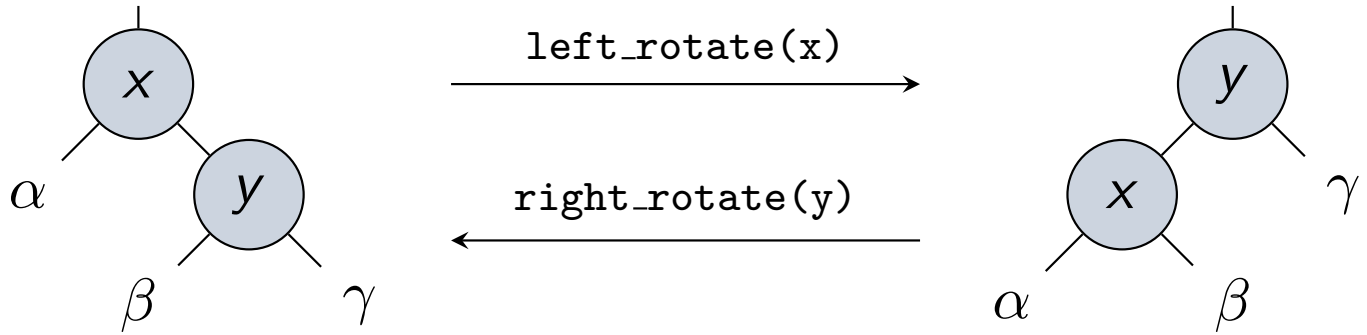
`https://www.cs.usfca.edu/~galles/visualization/RedBlack.html`

Insert (and delete) a number of keys into the red-black tree. What do you observe?

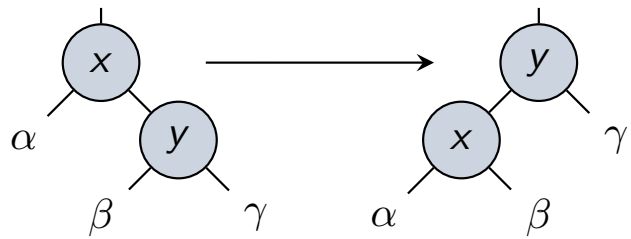
Rotation

- Inserting and deleting nodes as in binary search trees does not preserve the red-black property.
- Rotation is an operation that transforms the structure of the tree but preserves the binary-search-tree property.
- Two variants: left and right rotation.
- We use them to re-establish the red-black property during an insertion/deletion.



Left-Rotation

```
1  class RedBlackTree:
2      def __init__(self):
3          self.nil = Node(None, None, color=BLACK) # sentinel
4          self.root = self.nil
5
6      def left_rotate(self, x):
7          y = x.right
8          x.right = y.left
9          if y.left is not self.nil:
10             y.left.parent = x
11         y.parent = x.parent
12         if x.parent is self.nil: # x was root node
13             self.root = y
14         elif x is x.parent.left:
15             x.parent.left = y
16         else:
17             x.parent.right = y
18         y.left = x
19         x.parent = y
```



Insertion

```
1     def insert(self, key, value):
2         current = self.root
3         parent = self.nil
4         while current is not self.nil:
5             parent = current
6             if current.key > key:
7                 current = current.left
8             else:
9                 current = current.right
10        node = Node(key, value, color=RED)
11        node.parent = parent
12        if parent is self.nil: # tree was empty
13            self.root = node
14        elif key < parent.key:
15            parent.left = node
16        else:
17            parent.right = node
18        node.left = self.nil  # explicit leaf nodes
19        node.right = self.nil
20        self.fixup(node)
```

Up to this point pretty much like `insert` in binary search tree.

What red-black properties can be violated before the fixup?

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What could be violated before the fixup? Only 2 or 4!

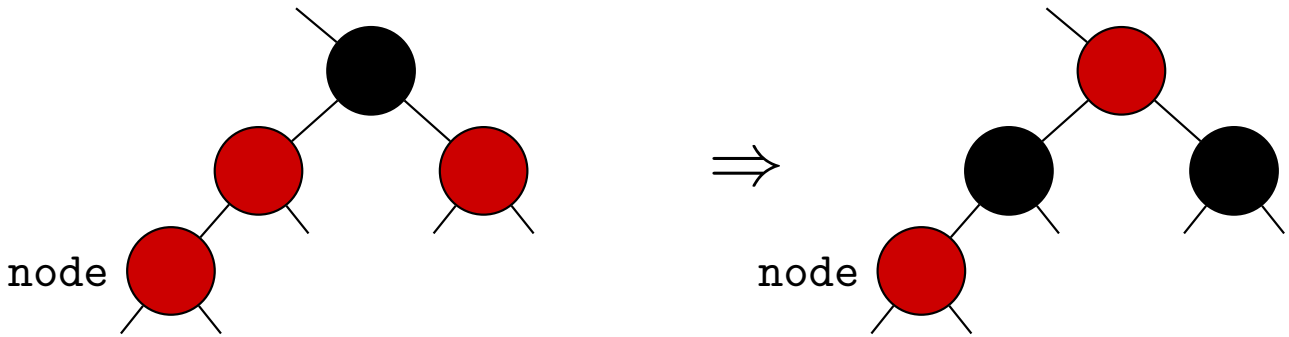
Property 2 is easy to re-establish: Just color the root black.
For property 4, distinguish three cases...

Fixup: Case 1

Potential problem: `node` and its parent are both red (the only violation of red-black property 4).

Case 1: The uncle (parent's sibling) of `node` is red.

- The grandparent of `node` cannot be red (by property 4).
- Idea: Make grandparent red and parent and uncle black.
- Afterwards: Need to fixup grandparent (its parent could be red).

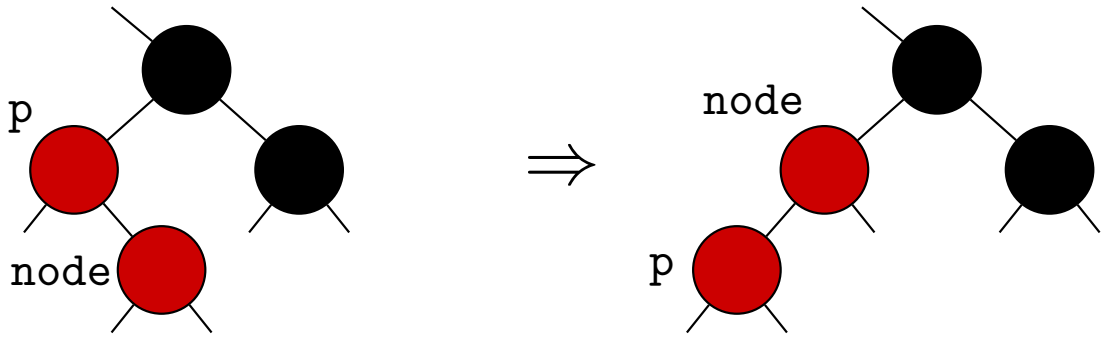


Fixup: Case 2

[Suppose `node`'s parent is a left child.]

Case 2: The uncle of `node` is black and `node` is a right child.

- Perform a left-rotation on the parent.
- Now the red previous parent is the left child of the red node.
- This constellation corresponds to case 3 (with the previous parent in the role of the red child node) and is resolved the same way (next slide).

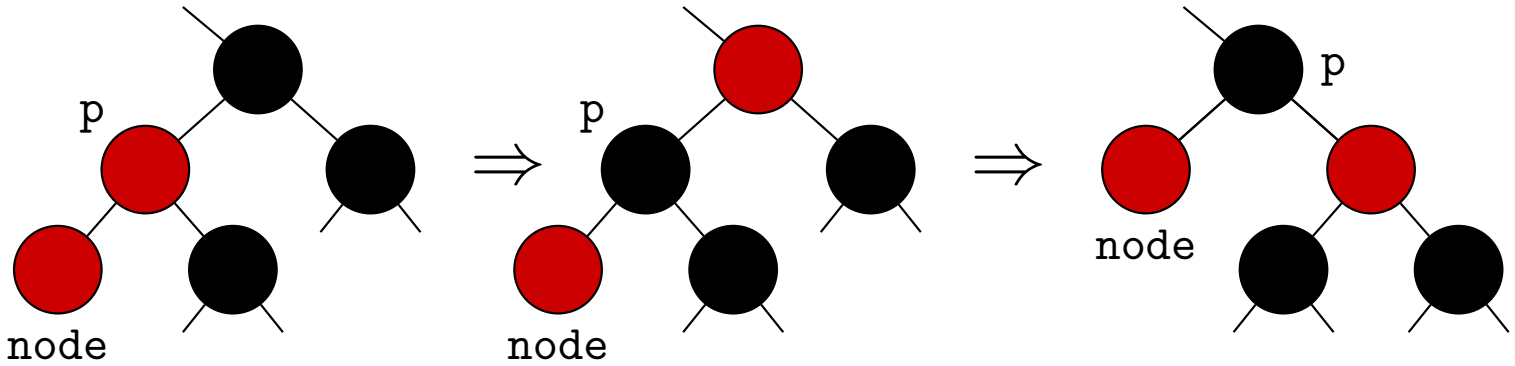


Fixup: Case 3

[Suppose `node`'s parent is a left child.]

Case 3: The uncle of `node` is black and `node` is a left child.

- Make parent black and grandparent red.
- Afterwards, perform a right-rotation on the grandparent.



Insertion: Fixup

```
 1      def fixup(self, node):
 2          while node.parent.color == RED:
 3              grandparent = node.parent.parent
 4              if node.parent is grandparent.left:
 5                  uncle = grandparent.right
 6                  if uncle.color == RED:
 7                      node.parent.color = BLACK
 8                      uncle.color = BLACK
 9                      grandparent.color = RED
10                      node = grandparent
11                  else:
12                      if node is node.parent.right:
13                          node = node.parent
14                          self.left_rotate(node)
15                      node.parent.color = BLACK
16                      node.parent.parent.color = RED
17                      self.right_rotate(grandparent)
18              else:
...                     ...
...                     # symmetric cases 1-3, where parent is
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Running time?

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Running time: $O(h)$
(h tree height)

Insertion: Running Time

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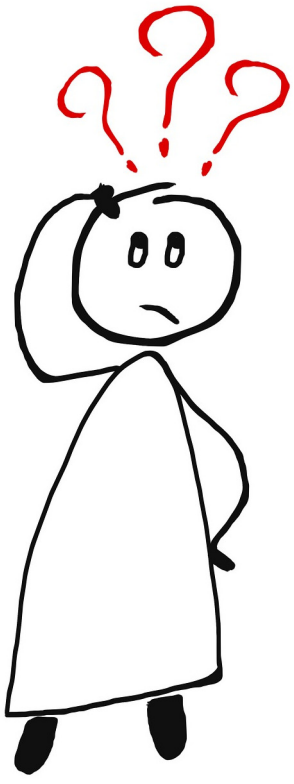

Questions?

Deletion

- Deleting a node from a red-black tree is more complicated than inserting a node.
- We do not cover the details in this course.
- Deletion from a tree with n nodes is possible in time $O(\log_2 n)$.

Summary

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- Red-black trees are a special kind of binary search trees that are approximately balanced.
- The height of a red-black tree with n nodes is $O(\log_2 n)$.
- Consequently, the query operations only take logarithmic time in the size of the tree.
- The same is true for insertion and deletion.