Algorithms and Data Structures

B3. Heaps, Priority Queues and Heapsort

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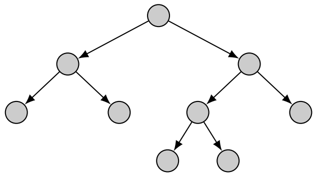
B3.1 Introduction

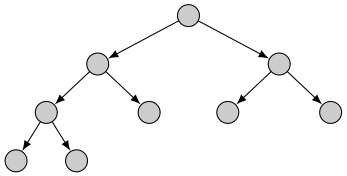
Our Plan for Today

- Data structure heap
- Algorithm heapsort that uses a heap.
- Abstract data type priority queue,
  that can be implemented with a heap.

B3.2 Heap

Binary Trees

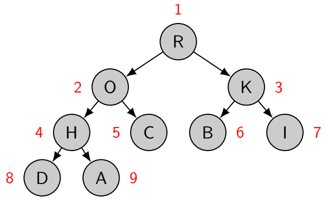




- Binary tree: each node has at most two successor nodes.
- We distinguish the left and the right child of a node.
- A single child can be the left or the right child.
- A nearly complete binary tree is completely filled on all levels except possibly the lowest, which is filled from left to right.

Nearly Complete Binary Trees as Arrays

- Consider 1-indexed arrays.
- Every such array can be interpreted as a nearly complete binary tree and vice versa.
  - Assign numbers $1, 2, \dots$ to nodes in tree from root to leaves and left to right on each level.
  - The number is the index in the array.
  - The left child of node i gets $2i$ and the right child $2i + 1$.



R	O	K	H	C	B	I	D	A
1	2	3	4	5	6	7	8	9

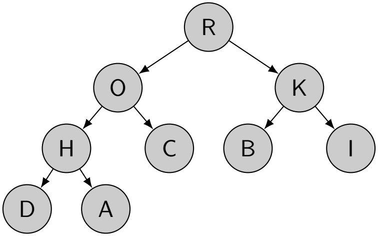
Helper Functions

```
def left(i):
    return 2 * i

def right(i):
    return 2 * i + 1

def parent(i):
    return i // 2
```

Heap: Max-Heap

Definition: Max-Heap

A nearly complete binary tree is a max-heap if the key stored in each node is greater or equal to the keys of each of its children.

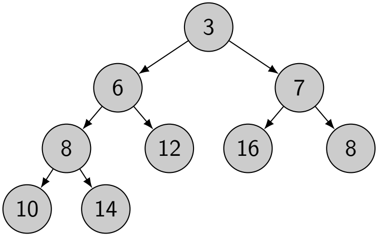


The largest key in a max-heap is at the root.

Heap: Min-Heap

Definition: Min-Heap

A nearly complete binary tree is a min-heap if the key stored in each node is smaller or equal to the keys of each of its children.



The smallest key in a min-heap is at the root.

We will focus on max-heaps. Min-heaps are implemented analogously.

Max-heaps: Operations

We will implement the following operations:

- `build_max_heap` transforms an array into a max-heap.
- `max_heap_maximum` returns the largest element.
- `max_heap_extract_max` removes and returns the largest element.
- `max_heap_insert` adds an item to the heap.

We will use two helper functions that fix local violations of the heap property:

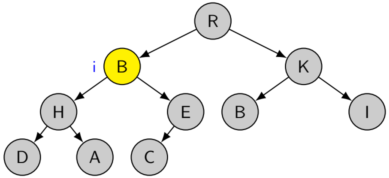
- `sink` moves an element with a too small key downwards.
- `swim` moves an element with a too large key upwards.

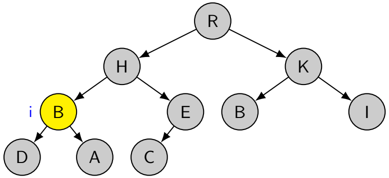
Helper Function: Sink

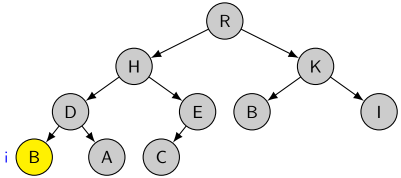
- `Sink` assumes that the left and right subtree of node i are max-heaps but the key at i might be smaller than the keys at $2i$ or $2i + 1$ (root of left and right sub-tree), violating the heap property.
- Idea: Let the entry recursively "float down" into the subtree with the larger key at its root.

In the book by Cormen et al. the function is called max_heapify.

Sink: Example







Jupyter Notebook

Jupyter notebook: `heaps.ipynb`

Sink: Implementation

```
def sink(heap, i, heap_size=None):
    if heap_size is None:
        heap_size = len(heap) - 1

    l = left(i)
    r = right(i)
    if l <= heap_size and heap[l] > heap[i]:
        largest = l
    else:
        largest = i
    if r <= heap_size and heap[r] > heap[largest]:
        largest = r
    if largest != i:
        heap[i], heap[largest] = heap[largest], heap[i]
        sink(heap, largest, heap_size)
```

Parameter `heap_size` can be used to exclude some entries at the end of the array from the heap (these positions will be ignored).

Sink: Running time

Simple insight:

- Let h be the height of the subtree rooted at position i.
- Then the worst-case running time of `sink` is $O(h)$.

Full story:

- Let n be the number of nodes of the subtree rooted at position i.
- Determining the final value of `largest` is $\Theta(1)$.
- Each subtree has size at most $2n/3$, so for the worst-case running time T of `sink`, we have

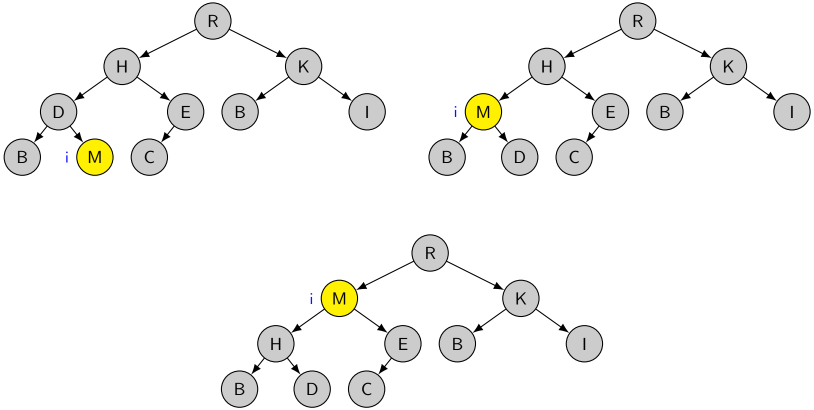
$$T(n) \leq T(2n/3) + \Theta(1).$$

- By master theorem (case 2), $T(n) \in O(\log_2 n)$.

Helper Function Swim

- `Sink` lets an entry with a too small key recursively "float down" into the subtree (a heap) with the larger key at its root.
- We now consider the counterpart `swim`: let an entry with a too large key float up in a tree that is otherwise a heap.

Swim: Example



Swim: Implementation

```
def swim(heap, i):
    parent_index = parent(i)
    # as long as i is not the root and the parent
    # of i has a smaller key than i
    while i > 1 and heap[parent_index] < heap[i]:
        # swap the entries of nodes i and its parent
        heap[parent_index], heap[i] = heap[i], heap[parent_index]

        # continue floating up the entry from the parent
        i = parent_index
        parent_index = parent(i)
```

Running time: $O(\log_2 n)$
(height of a nearly complete binary tree with n nodes is $\lfloor \log_2 n \rfloor$)

Build_max_heap

We can use sink to transform any array into a max-heap in a bottom-up fashion, processing all nodes from the second-lowest layer up to the root.

```
def build_max_heap(array):
    heap_size = len(array) - 1

    # all elements from positions heap_size//2 + 1
    # to heap_size are leaves of the tree.
    for i in range(heap_size//2, 0, -1):
        sink(array, i, heap_size)
```

Running Time of build_max_heap

- Heap with n elements has height $\lfloor \log_2 n \rfloor$.
- There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes rooting subtrees of height h.
  - The call of `sink` for each such node is $O(h)$.
  - Use c for the constant hidden in the asymptotic notation.

$$\begin{aligned}
T(n) &\leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil ch \\
&\leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h} ch = nc \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{h}{2^h} \\
&\leq nc \sum_{h=0}^{\infty} \frac{h}{2^h} \leq nc \frac{1/2}{(1-1/2)^2} \in O(n)
\end{aligned}$$

(cf. Cormen et al., p. 169 for reasons for inequalities; you may ignore the math.)

We can create a heap in linear time in the number of entries.

Determining the Maximum Element

In a max-heap, it is trivial to determine the largest element: it is the element at the root.

```
def max_heap_maximum(heap, heap_size):
    if heap_size < 1:
        raise Exception("empty heap")
    else:
        return heap[1]
```

Running time: $\Theta(1)$

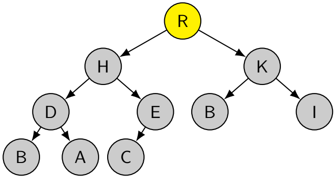
Extracting the Maximum Element

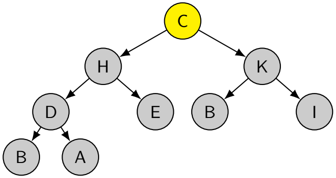
If we remove the largest element, we fill the position with the bottom-right element and restore the heap property with `sink` on position 1.

```
def max_heap_extract_max(heap, heap_size):
    maximum = max_heap_maximum(heap, heap_size)
    heap[1] = heap[heap_size]
    sink(heap, 1, heap_size)
    return maximum
    # the externally handled heap_size
    # needs to be decremented
```

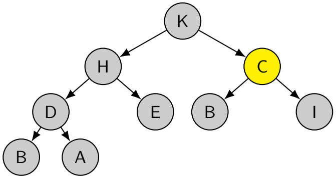
Running time: $O(\log_2 n)$ (with n size of the heap)

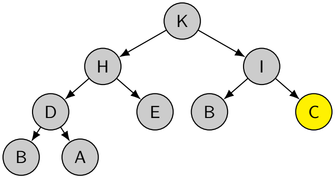
Extracting the Maximum Element: Example





Let the element sink from the root to a suitable node:





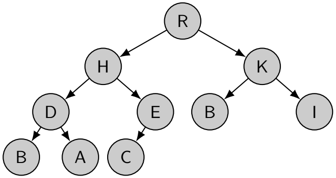
Inserting an Element

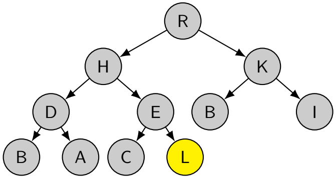
We insert an element as a new leaf and
let it swim to restore the heap property:

```
def max_heap_insert(heap, item, heap_size):
    if heap_size < len(heap) - 1:
        # we still have space in the array
        heap[heap_size + 1] = item
    else:
        assert heap_size == len(heap) - 1
        heap.append(item)
    swim(heap, heap_size + 1)
```

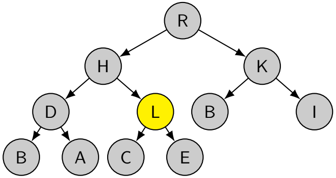
Running time: $O(\log_2 n)$ (with n size of the heap)
Only amortized if we are precise wrt. the append operation.

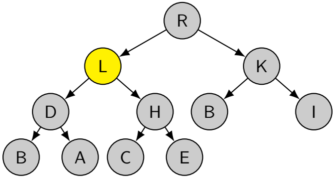
Inserting an Element: Example





Let the element swim from the leaf to a suitable node:





B3.3 Heapsort

Heapsort

- Basic idea as in selection sort but from right to left: Successively swap the largest element to the end of the non-sorted range.
- We can represent the heap directly in the input sequence, so that heapsort only needs constant additional memory.

Jupyter Notebook

Jupyter notebook: `heaps.ipynb`

Heapsort

```
# assumes that array[0] is not part of the input sequence
def heapsort(array):
    build_max_heap(array)
    # i ranges from last position down to position 1
    for i in range(len(array) - 1, 0, -1):
        # swap largest element from heap to position i
        array[i], array[1] = array[1], array[i]
        # restore heap_property for heap (in range 1,...,i-1)
        sink(array, 1, i-1)
```

- Building the heap takes linear time in n (length of array).
- We have a linear number of iterations of the for loop, each running in $O(\log_2 n)$.
- Overall running time $O(n \log_2 n)$.

Remarks

- Heapsort is asymptotically optimal wrt. running time and memory requirements:
  - Running time $O(n \log n)$.
  - Additional memory $O(1)$ (in-place)
- Practical disadvantage: Does not efficiently use the CPU cache because of poor locality of reference (swapping elements that do not have close storage locations)
- As an in-place approach still relevant, e.g. for embedded systems.

B3.4 Priority Queue

ADT Priority Queue

A priority queue is an ADT for maintaining a collection of elements, each with an associated key.

A max-priority queue supports the following operations:

- `insert(x, k)` inserts element `x` with key `k`.
- `maximum()` returns the element with the largest key.
- `extract_max()` returns and removes the element with the largest key.

Min-priority queues analogously prioritize elements with small keys.

Priority Queues: Applications

- Protocols for local area networks use them to ensure that high-priority applications experience lower latency than other applications.
- Prim's algorithm for minimum spanning trees and Dijkstra's algorithm for finding shortest paths in graphs use them for the processing order of the nodes of the graph (Ch. C4/C6).
- Huffman coding for lossless data compression uses them to prioritize nodes with high probability.

Jupyter Notebook

We can implement a priority queue with a heap:

Jupyter notebook: `heaps.ipynb`

B3.5 Summary

Summary

- (Max-)Heaps support the following operations:
  - Build heap from array: $O(n)$
  - Return largest element: $O(1)$
  - Remove largest element: $O(\log n)$
  - Insert element: $O(\log n)$
- Heapsort uses a heap to sort an array.
  - Can maintain the heap in the space of its input array.
  - In-place sorting algorithm.
- A priority queue is an abstract data type.
  - Can insert items with a priority ($=$ key).
  - Can obtain the item with the highest priority.
  - Implementation with heaps
    (or AVL trees or Fibonacci heaps; not covered in this course).