Algorithms and Data Structures

A13. Sorting: Lower Bound

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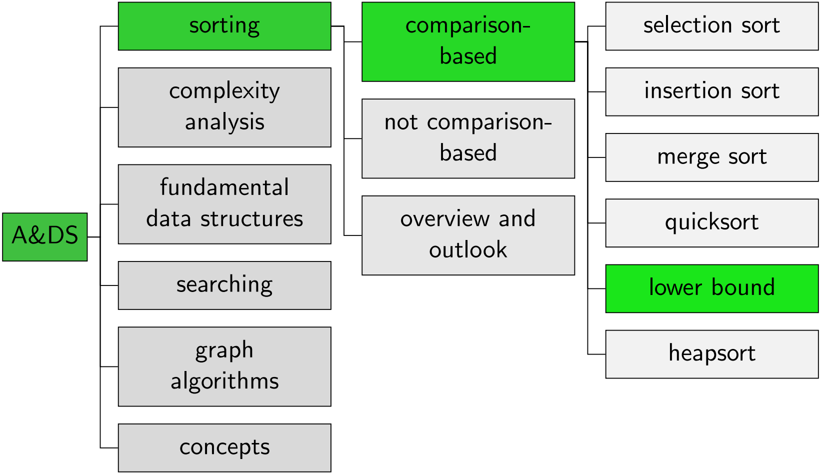
March 19, 2025

Lower Bound on Necessary Comparison Operations

Content of the Course



Question

- So far, merge sort and heapsort had with $O(n \log_2 n)$ the best (worst-case) running time.
- Can we do better?
- We show: Not with comparison-based approaches!

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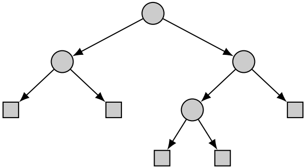
- Comparison-based approaches can only analyze the input by means of key comparisons.
- They must sort every input correctly.
- From this, we can derive a lower bound on the number of key comparisons in the worst case.

Crash Course: Binary Trees



- Binary tree: each node has at most two successor nodes.
- Nodes without successors are called leaves (squares in image).
- The node without a predecessor (at the top) is the root.
- The depth of a leaf is the number of edges from the root to the leaf.

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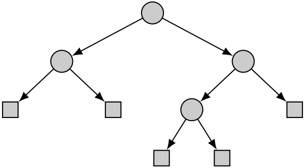

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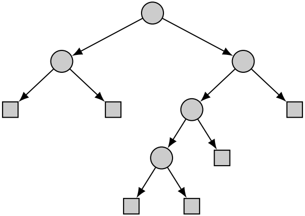
The maximal depth of a leaf in a binary tree with k leaves is at least $\log_2 k$.

Exercise (Slido)

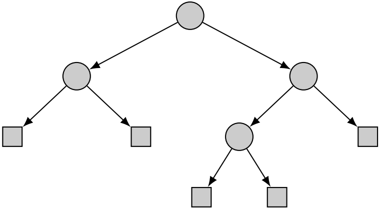


What is the maximal depth of a leaf in this tree?

Abstract Behavior as Tree

Consider an arbitrary comparison-based sorting algorithm A.

- Its behavior only depends on the results of key comparisons.
- For each key comparison, there are two possibilities how the algorithm proceeds.
- For an input of a given size, we can depict this graphically as a decision tree.



- Execution of A corresponds to tracing a simple path from the root down to a leaf.

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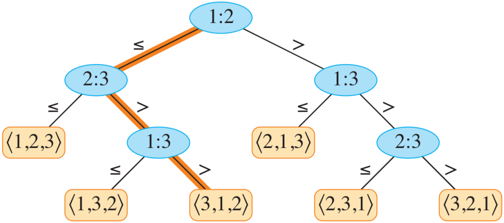
- Must sort all input sequences of size n correctly.
- We can adapt all algorithms so that they trace from which position to which position they move the elements.
- Then the result is not the sorted array, but the corresponding permutation.
- Since all possible inputs of size n must be sorted correctly, the algorithm must be able to generate all $n!$ possible permutations.

Example: Tree for Insertion Sort on Three Elements



Highlighted path e.g. for sorting sequence $[a_1 = 6, a_2 = 8, a_3 = 5]$

Source: Cormen et al., Introduction to Algorithms

Lower Bound I

- Each leaf in the tree corresponds to one permutation.
- For input size n, the tree must thus have at least $n!$ leaves.
- The maximal depth of a leaf in the tree is therefore $\geq \log_2(n!)$.
- There is an input of size n with $\geq \log_2(n!)$ key comparisons.

Lower Bound II

Lower bound on $\log_2(n!)$

- It holds that $n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$
  $4! = 1 \cdot 2 \cdot \underbrace{3}_{\geq 2} \cdot \underbrace{4}_{\geq 2} \geq 2^2$

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  $= \frac{n}{2}(\log_2 n + \log_2 \frac{1}{2}) = \frac{n}{2}(\log_2 n - \log_2 2)$
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Theorem

Every comparison-based sorting algorithm requires $\Omega(n \log n)$ key comparisons in the worst case. As a result, also the worst-case running time is $\Omega(n \log n)$.

Heapsort and merge sort are asymptotically optimal.

Summary

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- Every comparison-based sorting algorithm has at least linearithmic worst-case running time.